

# **YIELD LINE ANALYSIS OF SLABS WITH COVERED OPENINGS**

By

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## **ABSTRACT**

The yield line analysis method is used extensively throughout the blast design and analysis community to determine the strength of reinforced concrete structural elements to resist blast overpressures. Technical literature has long been available that describes the yield line method of analysis and explains how to derive the necessary analysis equations for a structural element such as a wall or a roof. Most presentations of the yield line analysis method deal with common slab configurations, such as one-way spans of varying support conditions or two-way spans that usually consist of a rectangular slab supported along two or more edges. Very little discussion is available on how to analyze slabs with openings, and even less information is available on how to analyze slabs with covered openings. Yet slabs with covered openings, such as blast-resistant doors or windows, are very common structural elements in the explosive safety design community. This paper presents a method by which the yield line analysis method can be used to analyze and design slabs and plates with openings.

## **EFFECT OF OPENINGS ON STRENGTH OF SLABS**

In blast resistant applications, the size of openings in slabs can become very significant when determining ultimate resistance to pressure. Openings tend to attract yield lines, but they don't automatically weaken a slab. Openings result in less surface area to collect load, and under some conditions, the ultimate strength of a slab can actually increase. When a slab has openings that are covered there is no reduction in surface area collecting load and the designer has to add into the analysis the effects of the additional blast load collected by the cover. These cover loads are passed into the slab via line loads around the supported edges of the cover. These line loads, if the cover is large enough or strong enough, can dramatically reduce the strength or ultimate resistance of the slab. Yield line analysis methods that include the effects of line loads can more accurately determine the strength of a uniform slab and often eliminate the need for pilasters and headers around the openings.

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## GENERAL PRINCIPLES OF YIELD LINE ANALYSIS

Yield line analysis is founded upon the principle of conservation of energy: the work performed by an external force moving through a distance is equal to the internal work performed by rotations about plastic hinges that resist the external force. The yield line analysis method provides an upper limit estimate of the maximum ultimate resistance,  $r_u$ , of a slab for an assumed mode of failure. The slab is assumed to fail by deflecting until plastic hinges form along the supports and in the interior of the slab. These plastic hinges or yield lines subdivide the slab into planar sectors that rotate about these hinges until maximum deflection or failure occurs. Any of several geometric combinations of plastic hinges can describe a valid mode for failure (failure mechanism) of a given slab, and each must be checked to determine which describes the lowest value of  $r_u$ . The failure mechanism that defines the lowest value of  $r_u$  will require the least amount of energy to fail. The assumption of an incorrect failure mechanism for a slab will result in either overpredicting the ultimate resistance of the element or in an undefined solution. This underscores the importance of checking all of the credible failure mechanisms for a slab to ensure the most reliable  $r_u$  has been identified.

There are two methods available in yield line analysis that can be used to calculate  $r_u$ : the virtual work method and the equilibrium method. Each method has its advantages and disadvantages as a design/analysis tool.

### VIRTUAL WORK METHOD

The virtual work method of yield line analysis makes use of the principle "if a rigid body that is in equilibrium under a system of forces, is given a virtual displacement, the sum of the virtual work done by the forces is zero" (Ref 1). A failure mechanism consisting of plastic hinges (yield lines) and planar sectors is assumed for a given slab. When an external pressure (blast load) is applied to the slab and it begins to deflect, yield lines begin to form. As the yield lines form, they become the axis of rotation of the planar sectors formed out of the slab. Each sector is assumed to act as a rigid body. As each sector rotates and deflects under the pressure, external and internal work is performed.

#### External Work

External work is represented by the external forces acting on the slab moving through the distance the slab deflects. The amount of external work performed is best calculated by computing the external forces acting on each rigid sector and multiplying them by the displacement of that sector. This is done by multiplying the pressure acting on the slab by the area of the sector, then multiplying that product by the distance the centroid of the area of the sector has displaced.

When a slab contains a covered opening, the cover over the opening resists the external overpressure and transfers that pressure to the slab via a line load along the cover supports. The external work performed by the line loads along a opening is computed by multiplying

the load times the length of the support to define a force. The force is then multiplied by the distance the line load moved. That distance can accurately be described as the deflection of the slab at the centroid of the line load.

The external work performed over the entire slab,  $W_{ext}$ , becomes the sum of the external work performed on each sector by the overpressure plus the sum of the work performed by any line loads around openings in the slab:

### EQUATION

$$W_{ext} = \sum r_u A_i \delta_a + \sum V L_j \delta_l \quad (1)$$

where:  $A_i$  = Area of Sector i  
 $\delta_a$  = Deflection at the centroid of the sector  
 $V$  = Line load per unit length  
 $L_j$  = Length of the line load j  
 $\delta_l$  = Deflection at the centroid of line load j

To determine how much overpressure load the slab can safely resist, the pressure acting on the slab is defined as the ultimate resistance,  $r_u$ , of the slab.

### Internal Work

As the sectors form and rotate, the ultimate moment capacity of the slab resists the bending moments, shear, and torsional forces along the yield lines. This is defined as internal work and can be computed by multiplying the ultimate moment capacity of the slab along each yield line (the ultimate moment capacity per unit length times the length of the yield line) by the angle of rotation along the yield line. The total internal work for a slab,  $W_{int}$ , is defined as the sum of the moment capacity along all yield lines:

## EQUATION

$$W_{int} = \sum M l_i \theta_i \quad (2a)$$

where:  $M$  = Ultimate moment capacity along yield line  
 $l_i$  = Length of yield line  $i$   
 $\theta_i$  = Rotation about yield line  $i$

The angle of rotation,  $\theta$ , can be approximated for small rotations by  $\tan \theta$ . In most cases where steel reinforcement is parallel and perpendicular to the supports, the internal work equation becomes

## EQUATION

$$W_{int} = \sum M_{ix} l_{iy} \theta_{iy} + \sum M_{iy} l_{ix} \theta_{ix} \quad (2b)$$

where:  $M_{ix}$  = Ultimate moment capacity along yield line  $i$  in  $x$  direction  
 $l_{iy}$  = Length of yield line  $i$  projected in the  $y$  direction  
 $\theta_{iy}$  = Rotation about yield line  $i$  projected about the  $y$  axis  
 $M_{iy}$  = Ultimate moment capacity along yield line  $i$  in  $y$  direction  
 $l_{ix}$  = Length of yield line  $i$  projected in the  $x$  direction  
 $\theta_{ix}$  = Rotation about yield line  $i$  projected about the  $x$  axis

The total external work is set equal to the total internal work for a slab. The resulting equation can then be solved for  $r_u$ . An assumed failure mechanism is considered to be invalid if the solution to the virtual work equation yields a negative value.

### Advantages and Disadvantages of Virtual Work

The advantage of the virtual work method is that it treats the entire slab as a unit which simplifies somewhat the derivation of the equations. Imbalances of shear and torsion forces along the yield lines cancel themselves out when forces are summed across the entire slab.

The disadvantage of the virtual work method is that the exact locations of the yield lines are

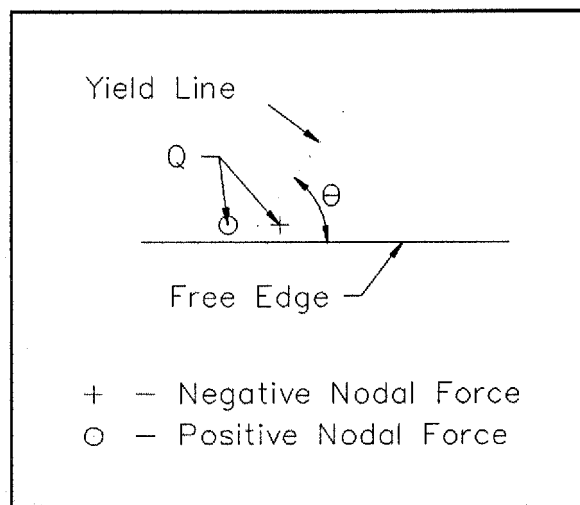
not known and they must be solved for. The equation derived to describe  $r_u$  for slabs with openings is large, complex, and usually nonlinear. The exact solution to the equation requires

partial differentiation with respect to each of the unknown locations of yield lines. The resulting equations are large, cumbersome and error prone. A trial and error solution where the absolute minimum  $r_u$  is computed based on assumed locations of the yield lines is an alternative solution method.

## EQUILIBRIUM METHOD

The equilibrium method is similar to the virtual work method, except that the external and internal work are computed and equated on a sector by sector basis. Imbalances of shear and torsional forces along the boundaries between sectors exist and must be accounted for in the equilibrium method. The adjustments for the imbalance of forces between sectors are known as nodal forces. Nodal forces in rectangular slabs with uniform reinforcement in each direction are significant only when yield lines intersect a free edge or an opening such as a door or window (Figure 1).

**Figure 1. Nodal Forces.**



**Figure 1. Nodal Forces.**

The expression for the nodal force,  $Q$ , is

### EQUATION

$$Q = \pm M_{\perp} \cot \theta$$

where:  $M_{\perp}$  = Moment capacity of steel  
perpendicular to free edge

$\theta$  = Acute angle formed by  
intersection of yield line with  
free edge

Note that when the yield line intersects a free edge at a 90-degree angle, the nodal force is zero. The nodal force is negative in the acute angle and positive in the obtuse angle.

An expression for  $r_u$  is derived for each sector under this method. A solution is reached when the locations of the yield lines are adjusted such that the value for  $r_u$  is the same for each of the sectors. The solution can be solved directly by equating the expressions for  $r_u$  from each sector and solving them as simultaneous equations. This effort is complex, as each relationship for  $r_u$  is, as with the virtual work method, usually a complex, nonlinear, algebraic expression.

An assumed failure mechanism is considered invalid when a simultaneous solution of the equilibrium equations yield a negative value for  $r_u$  or a yield line location. The failure mechanism is also invalid when the equations will not converge on a common value for  $r_u$  when using a trial and error solution.

### COMPARISON OF VIRTUAL WORK AND EQUILIBRIUM METHODS

Mathematical equations derived to solve for  $r_u$  of any particular failure mechanism theoretically yield basically the same result whether derived using the equilibrium method or the virtual work method. To assure accuracy and minimize potential errors, the equations for each failure mechanism described in this paper were derived and checked using both methods. Sample problems were run to verify that solutions from either method gave essentially the same result. In every failure case identified the results for  $r_u$  agreed within 1% or less. The locations of the yield lines at times, however, did vary significantly. In cases where differences did occur, the location of yield lines predicted by the equilibrium method were assumed to be the most correct because the work energy is balanced equally among all sectors.

The virtual work equation is somewhat simpler to derive, but is more complex to solve

directly. The equilibrium equations can be solved simultaneously, but involve more complex algebraic equations.

A trial and error solution, though it can be used for both methods, works best with the equilibrium method. As yield line locations are assumed, the computed value of  $r_u$  for each sector identifies trends that indicate which direction the assumed locations must be adjusted to achieve convergence. The logical identification of these "trends" work nicely with computer programming algorithms. The trial and error procedure is repeated until the designer is satisfied that the optimum locations of yield lines have been identified that yield a minimum value for  $r_u$ .

Because of the complex nature of the equations, yield line analysis solutions of slabs with covered openings are only practical when used in the form of a computer program.

## **REDUCTION OF MOMENT CAPACITY IN CORNERS**

Yield lines first begin to form in a structural element along the supports and on the interior of the slab. As deflection increases and a collapse mechanism begins to form, the yield lines extend and grow. Yield lines that form on the slab interior (positive yield lines) eventually extend to intersect with those that form along the supports (negative yield lines). At this point the collapse mechanism has fully formed. Experimental test results over the years clearly show that as yield lines extend into the corners of a slab, the localized stiffness prevents full rotations from developing, and the moment capacity of the slab in its corners is not fully developed. This causes the yield line analysis to overestimate  $r_u$  in these cases. Some design criteria recommend that reduction factors be applied to the computed  $r_u$  to more closely predict the actual value of the ultimate resistance (Ref 1, 2). The magnitude of the reduction factor varies according to the type of reinforcement and the geometry of the slab. The DOD design criterion (Ref 3) recommends that the moment capacity be reduced by 1/3 in corner regions to account for this reduction.

When a 1/3 reduction of moment capacity is accounted for in the derivation of yield line equations for slabs with openings, already complex equations become significantly more complex. For this reason, the yield line equations presented here were first derived without considering any reduction of slab moment capacity in the corners. Once agreement on  $r_u$  was reached between the two methods, the equations were expanded to account for the decreased capacity to absorb energy in the corners. Several important observations were noted when the 1/3 reduction of moment capacity was added to the equations. (1) The solution for  $r_u$  was consistently predicted to be 10-15% lower than that predicted without the reduction in moment capacity. (2) The solution for  $r_u$  using the virtual work method would not converge exactly with the equilibrium method. The virtual work equation consistently yielded a 1-2% lower value for  $r_u$ . (3) The virtual work method predicted significantly different locations for the yield lines. These phenomena were observed even in rectangular slabs without openings. The locations of the yield lines predicted by the equilibrium method were the same with or without a reduction of moment capacity in corners. The reason for these differences have not been determined, but they are most likely related to an inadvertent introduction of an



imbalance of equilibrium forces between internal and external work.

To minimize differences between solutions predicted by the two methods, and to simplify complex equations as much as possible, a singular reduction in  $r_u$  to account for corner effects is recommended as discussed in References 1 and 2. Table 1 shows recommended factors for rectangular slabs with right angle corners.

**Table 1. Recommended Reductions in Ultimate Resistance of Slab,  $r_u$ .**

Type of Slab	Reduction per Corner (%)		
	No Restraining Moments on Edges		Restrained Edges, Top Reinforced in Corners
	Top Reinforced in Corners <sup>a</sup>	No Top Reinforcement	
All Sides Supported	0	-1	-2
One Long Free Edge	-1	-3	-3

<sup>a</sup>Top steel area per unit width equal to bottom steel area per unit width.

**Table 1. Recommended Reductions in Ultimate Resistance of Slab,  $r_u$ .**

## YIELD LINE CONFIGURATIONS

The most likely failure mechanisms for slabs with openings along with the equations that have been derived for those mechanisms are given in Appendix A. For any slab with an opening, these failure patterns at a minimum should be investigated as possible collapse mechanisms. Simple geometrical considerations will eliminate most failure mechanisms for any given slab. Other possible mechanisms may exist and must be determined by following yield line theory and using good engineering judgment.

The configurations shown in Appendix A are designed to take advantage of symmetry for economy of space. If a slab does not exactly match the geometry shown in the figures, chances are it will match one of the cases if the orientation is rotated 90 degrees and/or reversed. For example, a slab with a door in the lower left corner can be reversed to match the opening in the lower right corner shown in Cases 1 through 4. If an opening in a slab is closer to a side wall than the floor, the failure pattern for the opening shown in Case 18 may apply by rotating the slab 90 degrees and adjusting the input to the equations to match the new rotated position. Similarly, slabs with openings should be rotated as many times as necessary to match up with as many of the failure mechanisms as possible to ensure that all possible cases have been checked. Software can easily be designed to perform these checks automatically.

The yield line equations shown in Appendix A assume that the slab has a covered opening. If a slab is to be analyzed that has no cover over the opening, the line load forces from the cover are simply input as zero. This will cause all of the line load terms to drop out.

### Modes of Failure

A slab with a covered opening can fail in one of two modes. The first and most common mode of failure is when the slab fails because of overpressure. The second mode of failure occurs when the slab fails primarily due to line loads from a cover over an opening. This mode of failure is not as common, but is important for the designer to be aware of.

A rigorous yield line analysis of a slab with a covered opening will then consist of two phases. The first phase will assume a failure mechanism based on overloading due to overpressure. The ultimate resistance for the slab is then computed, accounting for the effects of blast overpressure and line loads acting on the slab. The second phase assumes a yield line failure mechanism based on an overload from line loads acting on the wall. The ultimate resistance for the slab is then recalculated accounting for both the blast overpressure and the line loads on the slab based on that failure mechanism. The lowest value of  $r_u$  controls the design.

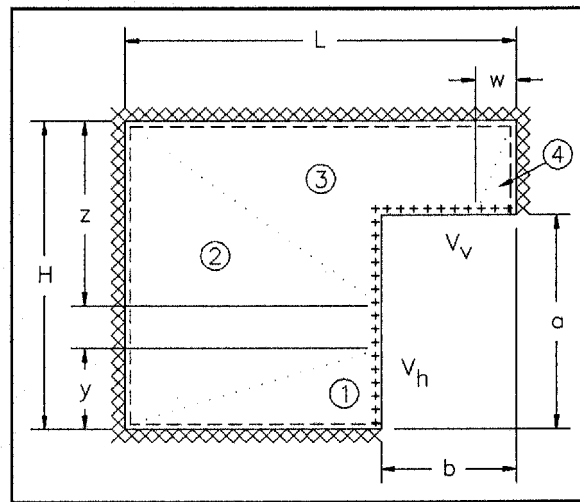
This paper documents only the first phase which represents the most common situations encountered in design and analysis of slabs with openings. To determine how much line loads influence the ultimate resistance, analyze the slab with and without line loads.

If the  $r_u$  with line loads is less than 50% of  $r_u$  without line loads, a phase 2 analysis should also be considered.

### **DERIVATION OF YIELD LINE EQUATIONS**

To perform a yield line analysis, a failure mechanism is first assumed for a slab. This paper shows the derivations of the equations used to describe the failure mechanism shown in Figure 2 (See Case 2 in Appendix A). This is a reinforced concrete slab of height,  $H$ , and length,  $L$ , that is fixed along all four edges, with a door of height,  $a$ , and width,  $b$ , located in the lower right corner. The door, while resisting a blast pressure will apply line loads,  $V_v$  across the top support, and  $V_h$  across the side support. Yield lines are identified by dashed lines. The variables  $w$ ,  $y$ , and  $z$  define the locations where yield lines intersect free edges of the doorway. The wall slab is divided by the yield lines into four sectors identified 1 - 4.

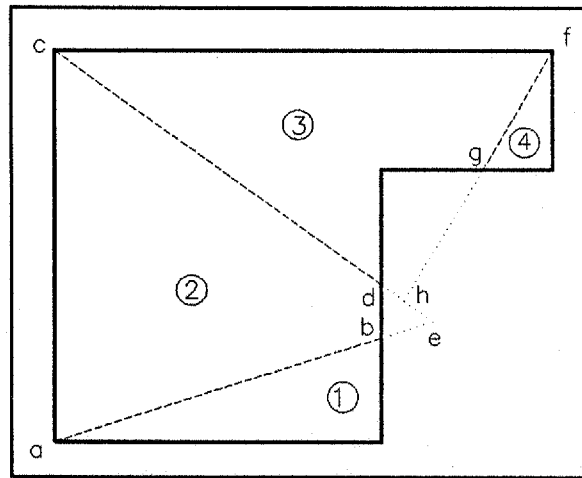
**Figure 2. Yield Line Failure Mechanism for Case 2.**



**Figure 2. Yield Line Failure Mechanism for Case 2.**

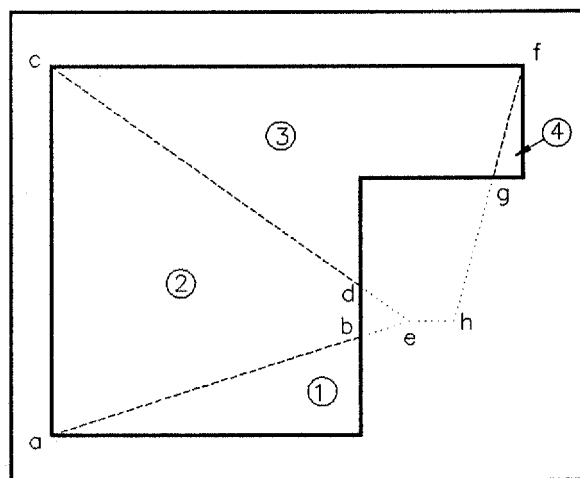
The yield line equations derived here equate the external work with the internal work as the structure deforms under a blast load. The sectors defined are assumed to be rigid bodies with planar rotation about its supports. Yield lines that form along the supports are called negative yield lines, and those forming in the span between sectors are called positive yield lines. The greatest displacement obtained by any one sector of the slab is defined as a unit displacement. All other displacements throughout the slab are normalized and expressed in terms of the unit displacement. If an opening is located on a portion of the slab where a maximum displacement would otherwise occur, the yield lines are extended into the opening until they intersect. The hypothetical intersection becomes the assumed location of the maximum, hence, the unit displacement.

**Figure 3. Extension of Yield Lines to Intersection.**



**Figure 3. Extension of Yield Lines to Intersection.**

**Figure 4. Horizontal Extension of Yield Lines from Point e.**



**Figure 4. Horizontal Extension of Yield Lines From Point e.**

For example, in Figure 3, Line a-b separates Sectors 1 and 2. Line c-d separates Sectors 2 and 3. These two lines are extended to intersect each other at Point e. The maximum or unit displacement is assumed to occur at Point e. Sectors 3 and 4 are separated by Line f-g. Line f-g must now be extended to intersect the positive yield Line c-e.

There are two possibilities of intersection. The first possibility is Line f-g will extend directly to intersect somewhere along Line c-e. To maintain planar rotation of Sector 3, Line f-g must be constrained to only intersect Line c-e between Points d and e. The other possibility is that the extension of Line f-g does not intersect Line c-e at all, but passes somewhere to the right of it. In this case a horizontal line must be extended out to the right from Point e. Line f-g is then extended to intersect the horizontal line. This defines Point h as shown in Figure 4. The displacement along Line e-h is defined as the unit displacement, and the rotations of all sectors as planar sections remain consistent.

Preliminary geometrical relationships that are needed in the derivation of the yield line equations are defined as illustrated in Figure 5 and as given below:

## EQUATION

$$w_2 = \frac{w L z}{(L - b)(H - a) + w z} \quad (\text{As shown in Figure 5a})$$

$$x_2 = \frac{H(L - b)}{y + z}$$

$$y_2 = \frac{H y}{y + z}$$

$$y_3 = \frac{H z}{y + z}$$

$$z_2 = \frac{L z (H - a)}{(L - b)(H - a) + w z}$$

(a)  $x_2 + w_2 > L$ .

(b)  $x_2 + w_2 \leq L$ .

For **Sector 1**, the equation for external work was derived using Equation 1 and is shown in Table 2(a). The internal work equation was derived using Equation 2a and is shown in Table 2(b). Setting  $W_{extl} = W_{intl}$  and solving for  $r_l$  yields the relationship shown in Table 2(c). The units used in these equation derivations are feet for distances and lengths, inch-pounds per inch for moments, psi for ultimate resistance, and pounds per foot for line loads. A conversion factor of 144 is used in all external work terms to keep units consistent.

Two equations are possible for Sectors 3 and 4, depending on the location of the yield lines. If  $x_2 + w_2 > L$  as shown in Figure 5(a), the equations are derived as follows.

For **Sector 4**, the external work is shown in Table 5(a), while the internal work is shown in Table 5(b). Setting  $W_{ext4} = W_{int4}$  and solving for  $r_4$  yields the equation shown in Table 5(c).

If  $x_2 + w_2 \leq L$ , the equations for Sectors 3 and 4 are derived from the geometry shown in Figure 5(b). The equations are derived as follows.

For **Sector 3**, the equation for external work is the same as shown in Table 4(a). However, the internal work is different as shown in Table 4(e). Setting  $W_{ext3} = W_{int3}$  and solving for  $r_3$  is shown in Table 4(f).

For **Sector 4**, the external work equation is shown in Table 5(d), while the internal work equation is shown in Table 5(e). Setting  $W_{ext4} = W_{int4}$  and solving for  $r_4$  yields Table 5(f).

The relationships derived for  $r_1$ ,  $r_2$ ,  $r_3$ , and  $r_4$  are all assumed to be equivalent ( $r_1 = r_2 = r_3 = r_4$ ). These equations may now be solved simultaneously or by trial and error methods.

Table 2. Yield Line Equations for Sector 1

<p>(a) External Work:</p> $W_{ext1} = 144 r_1 \left[ (L-b) y \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) \left( \frac{y}{y_2} \right) \right] + V_h \left( \frac{y}{2} \right) \left( \frac{y}{y_2} \right)$
<p>(b) Internal Work:</p> $W_{int1} = (M_{m1} + M_{vp}) \frac{L-b}{y_2} - M_{hp} \frac{y}{L-b}$
<p>(c) Ultimate Resistance:</p> $r_1 = \frac{M_{m1} \left( \frac{(L-b)(y+z)}{Hy} \right) + M_{vp} \left( \frac{(L-b)(y+z)}{Hy} \right) - M_{hp} \left( \frac{y}{L-b} \right) - V_h \left( \frac{y(y+z)}{2H} \right)}{24 \left( \frac{y(L-b)(y+z)}{H} \right)}$

Table 2. Yield Line Equations for Sector 1

Table 3. Yield Line Equations for Sector 2

<p>(a) External Work:</p> $W_{ext2} = 144 r_2 \left[ (L-b) \left( \frac{1}{2} \right) y \left( \frac{1}{3} \right) \frac{L-b}{x_2} + (H-y-z)(L-b) \left( \frac{1}{2} \right) \frac{L-b}{x_2} + \left( \frac{1}{2} \right) z(L-b) \left( \frac{1}{3} \right) \frac{L-b}{x_2} \right] \\ + V_h \frac{(H-y-z)(y+z)}{H}$
<p>(b) Internal Work:</p> $W_{int2} = M_{hnl} \frac{H}{x_2} + M_{hp} \frac{y+z}{x_2} + M_{hp} \frac{y}{L-b} + M_{hp} \frac{z}{L-b}$
<p>(c) Ultimate Resistance:</p> $r_2 = \frac{M_{hnl} \left( \frac{y+z}{L-b} \right) + M_{hp} \left[ \frac{(y+z)^2}{H(L-b)} + \frac{y+z}{L-b} \right] - V_h \left( \frac{(H-y-z)(y+z)}{H} \right)}{24 \left( \frac{(L-b)y(y+z)}{H} \right) + 72 \left( \frac{(H-y-z)(L-b)(y+z)}{H} \right) + 24 \left( \frac{z(L-b)(y+z)}{H} \right)}$

Table 3. Yield Line Equations for Sector 2



Table 4. Yield Line Equations for Sector 3.

If $x_2 + w_2 > L$ , then	
(a) External Work:	$W_{ext3} = 144 r_3 \left[ \left( \frac{1}{2} \right) (L-b) z \left( \frac{1}{3} \right) \left( \frac{z}{y_3} \right) + (b-w)(H-a) \left( \frac{1}{2} \right) \left( \frac{H-a}{y_3} \right) + \left( \frac{1}{2} \right) w(H-a) \left( \frac{1}{3} \right) \left( \frac{H-a}{y_3} \right) \right]$ $+ V_h \frac{(z-H+a)(H-a+z) \left( \frac{1}{2} \right)}{y_3} + V_v \frac{(b-w)(H-a)}{y_3}$
(b) Internal Work:	$W_{int3} = M_{vn2} \frac{L}{z_2} + M_{vp} \frac{(L-b+w)}{z_2} - M_{hp} \frac{z}{L-b} + M_{vp} \frac{w}{H-a}$
(c) Ultimate Resistance:	$r_3 = \frac{\left\{ M_{vn2} \left( \frac{(L-b)(H-a)+wz}{z(H-a)} \right) + M_{vp} \left[ \frac{(L-b+w)[(L-b)(H-a)+wz]}{Lz(H-a)} + \frac{w}{H-a} \right] \right\}}{24 \left( \frac{z(L-b)(y+z)}{H} \right) + 72 \left( \frac{(b-w)(H-a)^2(y+z)}{Hz} \right) + 24 \left( \frac{w(H-a)^2(y+z)}{Hz} \right)}$ $- M_{hp} \left( \frac{z}{L-b} \right) - V_h \left( \frac{(z-H+a)(H-a+z)(y+z)}{2Hz} \right) - V_v \left( \frac{(b-w)(H-a)(y+z)}{H-z} \right)$
If $x_2 + w_2 < L$ , then	
(d) External Work	Same as (a)
(e) Internal Work	$W_{int3} = M_{vn2} \left( \frac{L}{y_3} \right) + M_{vp} \left( \frac{L-b+w}{y_3} \right) - M_{hp} \left( \frac{z}{L-b} \right) + M_{vp} \left( \frac{w}{H-a} \right)$
(f) Ultimate Resistance	$r_3 = \frac{\left\{ M_{vn2} \left( \frac{L(y+z)}{Hz} \right) + M_{vp} \left[ \frac{(L-b+w)(y+z)}{Hz} + \frac{w}{H-a} \right] - M_{hp} \left( \frac{z}{L-b} \right) \right\}}{24 \left( \frac{z(L-b)(y+z)}{H} \right) + 72 \left( \frac{(b-w)(H-a)^2(y+z)}{Hz} \right) + 24 \left( \frac{w(H-a)^2(y+z)}{Hz} \right)}$ $- V_h \left( \frac{(z-H+a)(H-a+z)(y+z)}{2Hz} \right) - V_v \left( \frac{(b-w)(H-a)(y+z)}{Hz} \right)$

Table 4. Yield Line Equations for Sector 3.

Table 5. Yield Line Equations for Sector 4.

If $x_2 + w_2 > L$ , then	
(a) External Work:	$W_{ext4} = 144 r_4 \left( \frac{1}{2} \right) (H - a) w \left( \frac{1}{3} \right) \left( \frac{w}{w_2} \right) \left( \frac{z_2}{y_3} \right) + V_v w \left( \frac{1}{2} \right) \left( \frac{w}{w_2} \right) \left( \frac{z_2}{y_3} \right)$
(b) Internal Work:	$W_{int4} = (M_{hn2} + M_{hp}) \left( \frac{(H - a)[(L - b)(H - a) + wz]}{wLz} \right) - M_{vp} \frac{w}{H - a}$
(c) Ultimate Resistance:	$r_4 = \frac{(M_{hn2} + M_{hp}) \left( \frac{(H - a)[(L - b)(H - a) + wz]}{wLz} \right) - M_{vp} \left( \frac{w}{H - a} \right) - V_v \left( \frac{w(H - a)(z + y)}{2zH} \right)}{\left( \frac{24 w(H - a)^2 (y + z)}{Hz} \right)}$
If $x_2 + w_2 \leq L$ , then	
(d) External Work:	$W_{ext4} = 144 r_4 \left( \frac{1}{2} \right) (H - a) w \left( \frac{1}{3} \right) \left( \frac{w}{w_2} \right) + V_v w \left( \frac{1}{2} \right) \left( \frac{w}{w_2} \right)$
(e) Internal Work:	$W_{int4} = (M_{hn2} + M_{hp}) \left( \frac{H - a}{w_2} \right) - M_{vp} \frac{w}{H - a}$
(f) Ultimate Resistance:	$r_4 = \frac{(M_{hn2} + M_{hp}) \left( \frac{(H - a)^2 (y + z)}{wHz} \right) - M_{vp} \left( \frac{w}{H - a} \right) - V_v \left( \frac{w(H - a)(y + z)}{2Hz} \right)}{24 \left( \frac{w(H - a)^2 (y + z)}{Hz} \right)}$

Table 5. Yield Line Equations for Sector 4.

Equations were derived for Cases 1, 3-21 using similar procedures and are listed in Appendix A.

## STIFFNESS OF SLABS WITH OPENINGS

The stiffness of slabs with openings can have different levels of importance when computing the maximum deflection caused by a blast load. Stiffness is most important when expected maximum slab deflections will be in the elastic and elasto-plastic ranges. Stiffness becomes less important when maximum deflections extend more into the plastic deformation range where the ultimate resistance,  $r_u$ , increasingly influences the response. When maximum plastic deflections are large, the value for stiffness can be approximated with little loss of accuracy by computing the stiffness of a slab or plate as though there is no opening.

When greater accuracy is required in computing the stiffness of a slab with an opening, the following approach is recommended. Once a yield line analysis has been performed, a failure pattern identified, and the location of the yield lines determined, two equivalent slabs are created. The first equivalent slab (Slab 1) will have the same height, width, and support conditions as the original slab. Slab 1 will have a uniform moment capacity in the horizontal and vertical directions and for positive and negative bending. The uniform moment capacity is determined by averaging  $M_{hn1}$ ,  $M_{hn2}$ ,  $M_{hp}$ ,  $M_{vn1}$ ,  $M_{vn2}$ , and  $M_{vp}$ . The stiffness of this slab is computed using standard procedures and neglecting the effect of openings.

### Case 1 - Case 4

For Cases 1 - 4 shown Appendix A, compute the ratios  $a/H$  and  $b/L$ . If  $a/H$  is greater than  $b/L$ , then compute the stiffness of a second equivalent slab (Slab 2) fixed across the top, bottom and left side, with a free edge along the right side as shown in Figure 6. Use a height,  $H$ , and a length  $L-b$ . Use the uniform moment capacity for the equivalent slab computed above. Compute the difference between stiffness of the two slabs by subtracting the stiffness of Slab 2 from the stiffness of Slab 1. Multiply this difference by the ratio  $a/H$  and subtract the result from the stiffness of Slab 1.

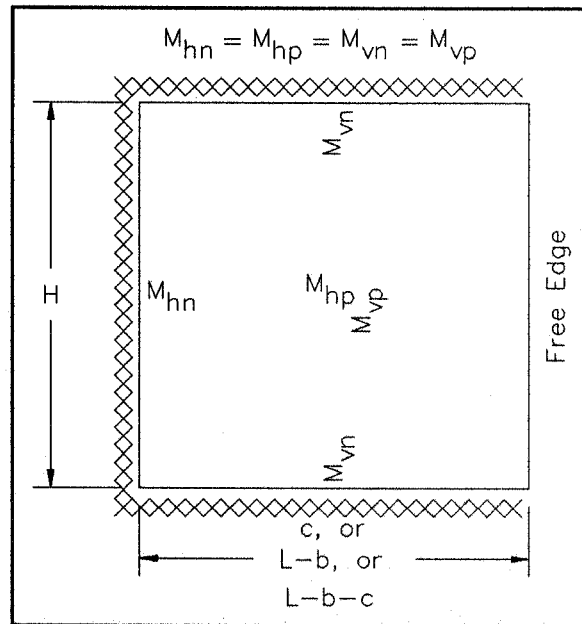
If  $a/H$  is less than  $b/L$ , then compute the stiffness of Slab 2 assuming the slab is fixed across the top, right and left sides, with a free edge along the bottom as shown in Figure 7. Use a height of  $H-a$  and a length  $L$ . Use the uniform moment capacity for Slab 1 computed above. Compute the difference between the stiffness of the two slabs by subtracting the stiffness of Slab 2 from Slab 1. Multiply this difference by the ratio  $b/L$  and subtract the result from the stiffness of Slab 1.

### Case 5 - Case 12

For Cases 5 - 12 shown in Appendix A, compute the ratios  $a/H$ ,  $c/L$ , and  $(L-b-c)/L$ . If  $a/H$  is greater than  $c/L$  and  $(L-b-c)/L$ , then compute the stiffness of equivalent Slab 2 assuming it is fixed across the top, bottom and left side, with a free edge along the right side as shown in Figure 6. Use a height  $H$  and a length  $c$  or  $L-b-c$ , whichever is larger. Use the uniform moment capacity for the equivalent slab computed above. Compute the difference between stiffness of the two slabs by subtracting the Slab 2 stiffness from Slab 1.

Multiply this difference by the ratio  $a/H$  and subtract the result from Slab 1.

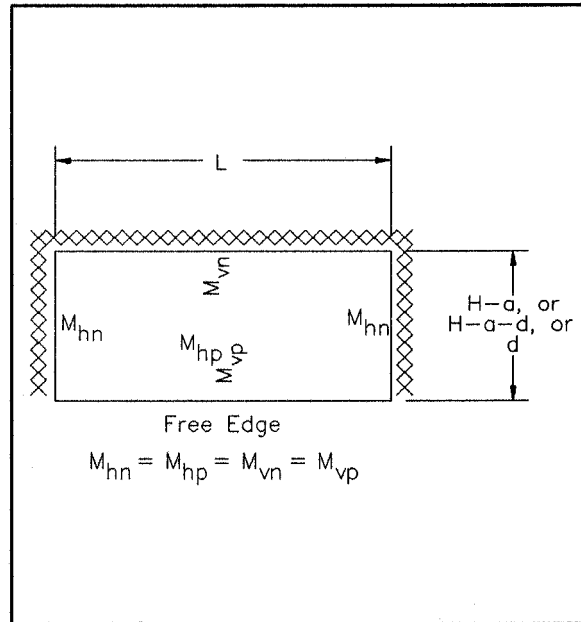
**Figure 6.**  
**Equivalent Slab 2 - Top, Bottom, and Left Sides Fixed, Right Edge Free.**



**Figure 6. Equivalent Slab 2 - Top,  
 Bottom, and Left Sides Fixed,  
 Right Edge Free.**

If  $a/H$  is less than  $c/L$  or  $(L-b-c)/L$ , then compute the stiffness of equivalent Slab 2 fixed across the top, right and left sides, with a free edge along the bottom as shown in Figure 7. Use a height of  $H-a$  and a length  $L$ . Use the uniform moment capacity for Slab 1 computed above. Compute the difference between stiffness of the two slabs by subtracting the Slab 2 stiffness from the Slab 1 stiffness. Multiply this difference by the greater of the two ratios  $c/L$  or  $(L-b-c)/L$  and subtract the result from the stiffness of Slab 1.

**Figure 7. Equivalent Slab 2 - Top, and Sides Edges Fixed, Bottom Edge Free.**



**Figure 7. Equivalent Slab 2 - Top  
and Side Edges Fixed, Bottom  
Edge Free.**

#### Case 13 - Case 21

For cases 13 - 21 shown in Appendix A, compute the ratios  $d/H$ ,  $(H-a-d)/H$ ,  $c/L$ , and  $(L-b-c)/L$ . Ratio 1 is the larger of the two ratios  $d/H$  and  $(H-a-d)/H$ . Ratio 2 is the larger of the two ratios  $c/L$  and  $(L-b-c)/L$ .

If Ratio 1 is less than Ratio 2, then compute the stiffness of equivalent Slab 2 assuming it is fixed across the top, bottom and left side, with a free edge along the right side as shown in Figure 6. Use a height  $H$  and a length  $c$  or  $L-b-c$ , whichever is larger. Use the uniform moment capacity for Slab 1 computed above. Compute the difference between stiffness of the two slabs by subtracting the Slab 2 stiffness from the Slab 1 stiffness. Multiply this difference by the ratio  $a/H$ . Subtract the result from the stiffness of Slab 1.

If Ratio 1 is greater than Ratio 2, then compute the stiffness of equivalent Slab 2 fixed across the top, right and left sides, with a free edge along the bottom as shown in Figure 7. Use a height of  $d$  or  $H-a-d$ , whichever is greater, and a length  $L$ . Use the uniform moment capacity for Slab 1 computed above. Compute the difference between the two stiffnesses by subtracting the Slab 2 stiffness from the Slab 1 stiffness. Multiply this difference by the ratio  $b/L$ . Subtract the result from the stiffness of Slab 1.

## SUMMARY

New design aids for yield line analysis of slabs with covered openings have been presented. Because of the complex nature of yield line equations, the analysis is best handled via computer programs. FORTRAN subroutines for each of the failure mechanisms presented in this paper are currently under development at the Naval Facilities Engineering Service Center. This work is being performed as part of the Defense Nuclear Agency DAHS-CWE project. The yield line equations shown here were derived using the equilibrium method. For the software that is under development, the equilibrium method is being used to predict the locations of the yield lines. The virtual work method is also being used to compute the ultimate resistance of the slab as it tends to be slightly more conservative.

## SAMPLE PROBLEM

Compute the ultimate resistance and stiffness of a rectangular slab with a door as shown in Figure 8. The wall is 36 inches thick and contains #8 steel reinforcing bars at 12 inches on center in the vertical direction and #9 steel reinforcing bars at 12 inches on center in the horizontal direction. The door is constructed of a 2-inch-thick A-36 steel plate supported across the top and two sides and free spanning across the bottom. The wall will be allowed up to 8 degrees of support rotation due to blast loading.

**Figure 8. Wall with Covered Door Opening, Sample Problem.**

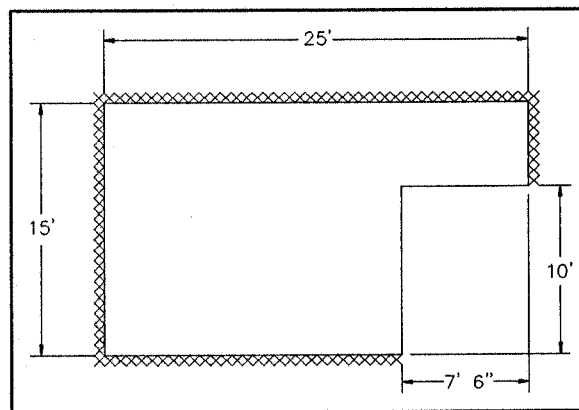


Figure 8. Wall With Covered  
Door Opening, Sample Problem.

## SOLUTION:

STEP 1. Compute the moment capacity of the walls

The moment capacity is calculated using standard procedures outlined in Reference 3.

$$\begin{array}{ll} M_{hn1} = 187,393 \text{ in.-lbs/in.} & M_{vn1} = 238,141 \text{ in.-lbs/in.} \\ M_{hn2} = 187,393 \text{ in.-lbs/in.} & M_{vn2} = 238,141 \text{ in.-lbs/in.} \\ M_{hp} = 187,393 \text{ in.-lbs/in.} & M_{vp} = 238,141 \text{ in.-lbs/in.} \end{array}$$

STEP 2. Compute the ultimate resistance of the door and corresponding line loads on the wall.

From Reference 3,  $r_u$  of the door is computed to be 72.0 psi. The intersection of the yield lines occur at  $x = 3.75$  feet,  $y = 5.11$  feet. From these tributary areas,

### EQUATIONS

$$V_h = \frac{\left[ \left( \frac{10' + 5.11'}{2} \right) (3.75') (72 \text{ psi} * 12 * 12) \right]}{10'} = 29,375 \text{ lbs/ft}$$

$$V_v = \left( \frac{1}{2} \right) 7.5' (10' - 5.11') (72 \text{ psi} * 12 * 12) = 25,345 \text{ lbs/ft}$$

STEP 3. Compute the ultimate resistance,  $r_u$ , of the slab.

Solve for  $r_u$  of the slab for each of Cases 1-4 in Appendix A. Use the smallest value of  $r_u$  obtained from the four cases. The results of a trial and error solution for Cases 1-4 are summarized below:

**TABLE 6**

Case Number	Ultimate Resistance, $r_u$
1	189.1 psi
2	did not converge
3	did not converge
4	did not converge

STEP 4. Compute the average moment capacity of equivalent Slab 1.

**EQUATION**

$$M_{avg} = \frac{3(187,393) + 3(238,141)}{6} = 212,767 \text{ in.-lb/in.}$$

STEP 5. Compute the stiffness of Slab 1.

From procedures in Reference 3:  $K_e = 2053.9$  psi-in.

STEP 6. Compute the ratios a/H, b/L; Determine configuration of Slab 2.

$$a/H = 10 \text{ ft}/15 \text{ ft} = 0.66667 \text{ ft} \quad b/L = 7.5 \text{ ft}/25 \text{ ft} = 0.30000 \text{ ft}$$

a/H is greater than b/L. Therefore, Slab 2 will be fixed along top, bottom, and left side, with a height of 15 feet and a width of 17.5 feet.  $M_{avg} = 212,767$  as before.

STEP 7. Compute the stiffness of Slab 2.

From procedures in Reference 3:  $K_e = 1223.5$  psi-in.

STEP 8. Compute the difference between the two stiffnesses.

$$K_e (\text{Slab 1}) - K_e (\text{Slab 2}) = 2053.9 - 1223.5 = 830.4 \text{ psi-in.}$$



STEP 9. Multiply the difference by the ratio a/H.

$$830.4 (a/H) = 830.4 (0.66667) = 553.6 \text{ psi-in.}$$

STEP 10. Subtract the result from the stiffness of Slab 1.

$$R_u (\text{Slab with opening}) = 189.1 \text{ psi}$$

$$K_e (\text{Slab with opening}) = 2053.9 - 553.6 = 1500.3 \text{ psi-in.}$$

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3. Structures to Resist the Effects of Accidental Explosions, Army TM 5-1300, Navy NAVFAC P-397, Air Force AFR 88-22, Dept of Defense Explosives Safety Board, Washington, D.C., 1990
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5. MacGregor, Reinforced Concrete - Mechanics and Design, Prentice Hall, Englewood Cliffs, NJ, 1988, 799 pp.
6. S. Islam, R. Park, Yield-line Analysis of Two Way Reinforced Concrete Slabs With Openings., The Structural Engineer, No. 6, Vol. 49, June 1971, pp. 269-276.
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## TABLE OF ABBREVIATIONS

### Geometric Terms

H	=	Height of the slab, ft.
L	=	Width of the slab, ft.
a	=	Height of opening, ft.
b	=	Width of opening, ft.
c	=	Distance from opening to right side wall, ft.
d	=	Height of opening above floor, ft.

### Structural Properties

$M_{hn1}$	=	Negative moment capacity along horizontal support No. 1, in.-lb/in.
$M_{hn2}$	=	Negative moment capacity along horizontal support No. 2, in.-lb/in.
$M_{hp}$	=	Positive horizontal moment capacity within the slab, in.-lb/in.
$M_{vn1}$	=	Negative moment capacity along vertical support No. 1, in.-lb/in.
$M_{vn2}$	=	Negative moment capacity along vertical support No. 2, in.-lb/in.
$M_{vp}$	=	Positive vertical moment capacity within the slab, in.-lb/in.
$r_u$	=	Ultimate resistance of a slab, psi.
$r_1$	=	Ultimate resistance of Sector 1 of a slab, psi.
$r_2$	=	Ultimate resistance of Sector 2 of a slab, psi.
$r_3$	=	Ultimate resistance of Sector 3 of a slab, psi.
$r_4$	=	Ultimate resistance of Sector 4 of a slab, psi.
$r_5$	=	Ultimate resistance of Sector 5 of a slab, psi.
$V_h$	=	Horizontal line loads along either side of opening, lb/ft.
$V_v$	=	Vertical line loads along top and/or bottom of opening, lb/ft.

### Yield Line Terms

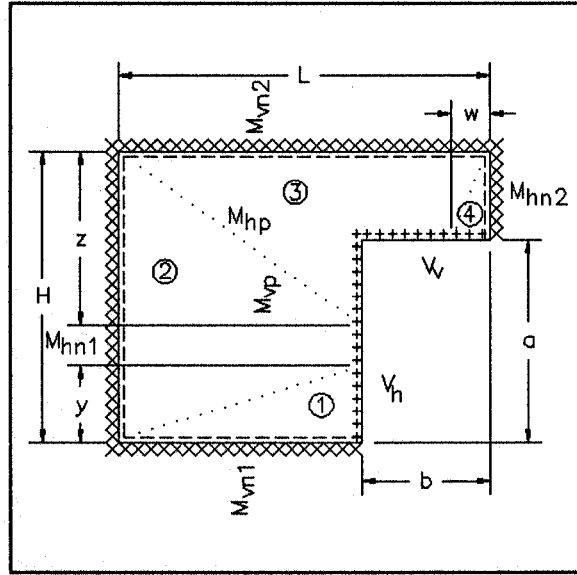
u	=	Location of yield line, ft.
v	=	Location of yield line, ft.
w	=	Location of yield line, ft.
x	=	Location of yield line, ft.
y	=	Location of yield line, ft.
z	=	Location of yield line, ft.

**APPENDIX A**

**YIELD LINE FAILURE MECHANISMS AND EQUATIONS**

**CASES 1 THROUGH 21**

$$r_4 = \frac{M_{bz2} \left( \frac{(H-a)^2}{w(H-y)} \right) + M_{bp} \left( \frac{(H-a)^2}{w(H-y)} \right) - M_{vp} \left( \frac{w}{H-a} \right) - V_v \left( \frac{w(H-a)}{2(H-y)} \right)}{24 \left( \frac{w(H-a)^2}{H-y} \right)}$$



Case 2

$$r_1 = \frac{M_{vn1} \left( \frac{(L-b)(y+z)}{Hy} \right) + M_{vp} \left( \frac{(L-b)(y+z)}{Hy} \right) - M_{hp} \left( \frac{y}{L-b} \right) - V_h \left( \frac{y(y+z)}{2H} \right)}{24 \left( \frac{y(L-b)(y+z)}{H} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{y+z}{L-b} \right) + M_{hp} \left[ \frac{(y+z)^2}{H(L-b)} + \frac{y+z}{L-b} \right] - V_h \left( \frac{(H-y-z)(y+z)}{H} \right)}{24 \left( \frac{(L-b)y(y+z)}{H} \right) + 72 \left( \frac{(H-y-z)(L-b)(y+z)}{H} \right) + 24 \left( \frac{z(L-b)(y+z)}{H} \right)}$$

When  $\frac{H(L-b)}{y+z} + \frac{wLz}{(L-b)(H-a) + wz} \leq L$ , then:

$$r_3 = \frac{\left\{ M_{vn2} \left( \frac{L(y+z)}{Hz} \right) + M_{vp} \left[ \frac{(L-b+w)(y+z)}{Hz} + \frac{w}{H-a} \right] - M_{hp} \left( \frac{z}{L-b} \right) \right\} - V_h \left( \frac{(z-H+a)(H-a+z)(y+z)}{2Hz} \right) - V_v \left( \frac{(b-w)(H-a)(y+z)}{Hz} \right)}{24 \left( \frac{z(L-b)(y+z)}{H} \right) + 72 \left( \frac{(b-w)(H-a)^2(y+z)}{Hz} \right) + 24 \left( \frac{w(H-a)^2(y+z)}{Hz} \right)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{(H-a)^2(y+z)}{wHz} \right) + M_{hp} \left( \frac{(H-a)^2(y+z)}{wHz} \right) - M_{vp} \left( \frac{w}{H-a} \right) - V_v \left( \frac{w(H-a)(y+z)}{2Hz} \right)}{24 \left( \frac{w(H-a)^2(y+z)}{Hz} \right)}$$

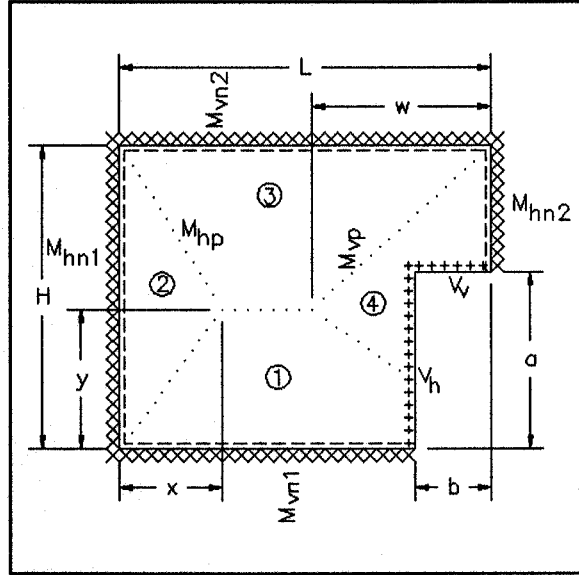
When  $\frac{H(L-b)}{y+z} + \frac{wLz}{(L-b)(H-a) + wz} > L$ , then:

$$r_3 = \frac{\left\{ M_{vn2} \left( \frac{(L-b)(H-a) + wz}{z(H-a)} \right) + M_{vp} \left[ \frac{(L-b+w)[(L-b)(H-a) + wz]}{Lz(H-a)} + \frac{w}{H-a} \right] \right.}{- M_{hp} \left( \frac{z}{L-b} \right) - V_h \left( \frac{(z-H+a)(H-a+z)(y+z)}{2Hz} \right) - V_v \left( \frac{(b-w)(H-a)(y+z)}{Hz} \right)} \Bigg\}$$

$$24 \left( \frac{z(L-b)(y+z)}{H} \right) + 72 \left( \frac{(b-w)(H-a)^2(y+z)}{Hz} \right) + 24 \left( \frac{w(H-a)^2(y+z)}{Hz} \right)$$

$$r_4 = \frac{\left\{ M_{hn2} \left( \frac{(H-a)[(L-b)(H-a) + wz]}{wLz} \right) + M_{hp} \left( \frac{(H-a)[(L-b)(H-a) + wz]}{wLz} \right) \right.}{- M_{vp} \left( \frac{w}{H-a} \right) - V_v \left( \frac{w(H-a)(z+y)}{2zH} \right)} \Bigg\}$$

$$24 \left( \frac{w(H-a)^2(z+y)}{Hz} \right)$$



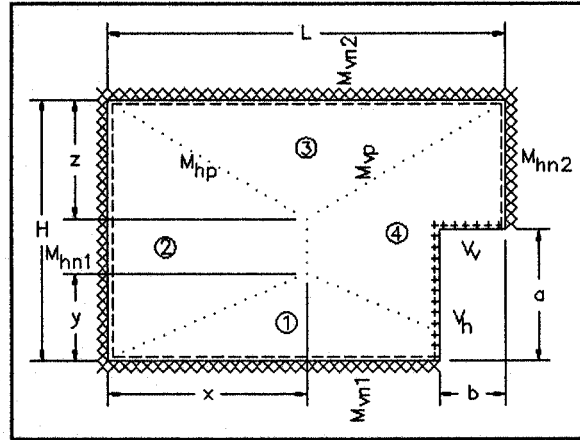
Case 3

$$r_1 = \frac{M_{vn1} \left( \frac{L-b}{y} \right) + M_{vp} \left( \frac{L-b}{y} \right) + M_{hp} \left( \frac{y}{w} \right) - V_h \left( \frac{b^2 y}{2 w^2} \right)}{24 x y + 24 w y + 72 (L-w-x) y + 24 \left( \frac{b^3 y}{w^2} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24 h x}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L}{H-y} \right)}{24 x (H-y) + 24 w (H-y) + 72 (H-y) (L-w-x)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H-a}{w} \right) + M_{hp} \left[ \frac{H-y}{w} - \frac{b y}{w^2} \right] - V_h b \left( \frac{a}{w} - \frac{b y}{w^2} \right) - V_v \left( \frac{b^2}{2 w} \right)}{24 H w - 72 b^2 \left( \frac{a}{w} - \frac{b y}{w^2} \right) - 24 \left( \frac{b^3 y}{w^2} \right)}$$



Case 4

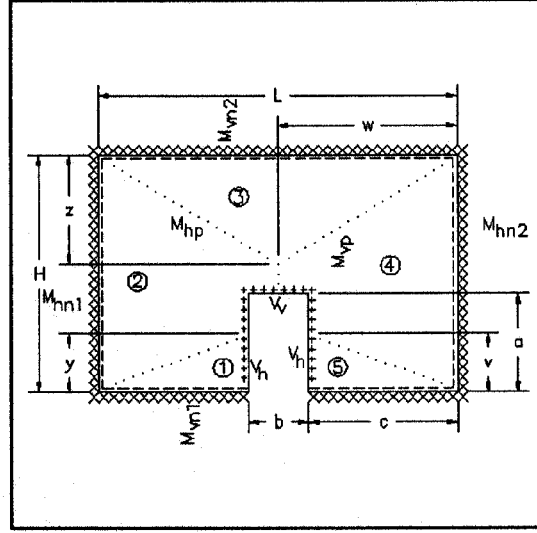
$$r_1 = \frac{M_{vn1} \left( \frac{L-b}{y} \right) + M_{vp} \left( \frac{L-b}{y} \right) + M_{hp} \left( \frac{y}{L-x} \right) - V_h \left( \frac{b^2 y}{2(L-x)^2} \right)}{24 L y - 24 \left( \frac{b^3 y}{(L-x)^2} \right)}$$

$$r_2 = \frac{M_{hn2} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24 x y + 24 z x + 72 (H-y-z) x}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{z} \right) + M_{vp} \left( \frac{L}{z} \right)}{24 L z}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H-a}{L-x} \right) + M_{hp} \left[ \frac{H-y}{L-x} - \frac{b y}{(L-x)^2} \right] - V_h b \left( \frac{a}{L-x} - \frac{b y}{(L-x)^2} \right) - V_v \left( \frac{b^2}{2(L-x)} \right)}{24 z (L-x) + 24 y (L-x) + 72 (L-x) (H-y-z) - 72 b^2 \left[ \frac{a}{L-x} - \frac{b y}{(L-x)^2} \right] - 24 \left( \frac{b^3 y}{(L-x)^2} \right)}$$





Case 5

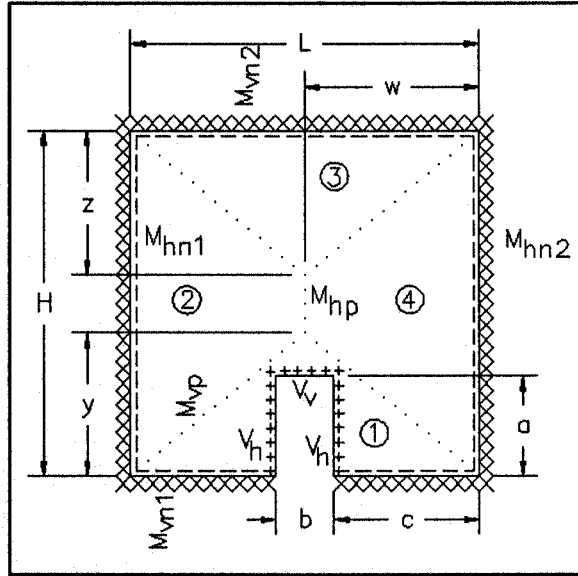
$$r_1 = \frac{M_{vn1} \left( \frac{(L-b-c)^2}{y(L-w)} \right) + M_{vp} \left( \frac{(L-b-c)^2}{y(L-w)} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) - V_h \left( \frac{y(L-b-c)}{2(L-w)} \right)}{24 \left( \frac{(L-b-c)^2 y}{L-w} \right)}$$

$$r_2 = \frac{\left\{ M_{hn1} \left( \frac{H}{L-w} \right) + M_{hp} \left[ \frac{H-a+y}{L-w} + \frac{y}{L-b-c} \right] - V_h \left( \frac{(a-y)(L-b-c)}{L-w} \right) - V_v \left( \frac{(b+c-w)(2L-b-c-w)}{2(L-w)} \right) \right\}}{24 \left( \frac{(L-b-c)^2 y}{L-w} \right) + 72 \left( \frac{(a-y)(L-b-c)^2}{L-w} \right) + 72 (H-z-a)(L-w) + 24 z(L-w)}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{z} \right) + M_{vp} \left( \frac{L}{z} \right)}{24 L z}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left[ \frac{H-a+v}{w} + \frac{v}{c} \right] - V_h \left( \frac{(a-v)c}{w} \right) - V_v \left( \frac{(w-c)(w+c)}{2w} \right)}{24 w z + 72 w (H-z-a) + 72 \left( \frac{c^2(a-v)}{w} \right) + 24 \left( \frac{c^2 v}{w} \right)}$$

$$r_5 = \frac{M_{vn1} \left( \frac{c^2}{v w} \right) + M_{vp} \left( \frac{c^2}{v w} \right) - M_{hp} \left( \frac{v}{c} \right)}{24 \left( \frac{c^2 v}{w} \right)}$$



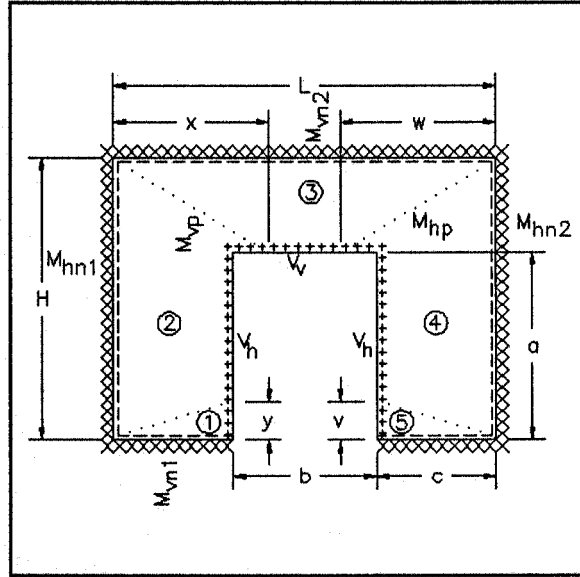
Case 6

$$r_1 = \frac{M_{vn} \left( \frac{L-b}{y} \right) + M_{vp} \left( \frac{L}{y} \right) - V_h \left( \frac{a^2}{y} \right) - V_v \left( \frac{ba}{y} \right)}{24Ly - 72 \left( \frac{ba^2}{y} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{L-w} \right) + M_{hp} \left( \frac{H}{L-w} \right)}{24y(L-w) + 24z(L-w) + 72(H-y-z)(L-w)}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{z} \right) + M_{vp} \left( \frac{L}{z} \right)}{24Lz}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right)}{24wz + 24wy + 72w(H-y-z)}$$



Case 7

When  $H - \frac{L(H-a)}{x+w} \geq \frac{y \times L}{(x+w)(L-b-c)}, \frac{v \times L}{c(x+w)}$ , then:

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{(L-b-c)^2(x+w)}{y \times L} \right) + M_{vp} \left( \frac{(L-b-c)^2(x+w)}{y \times L} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) \right.}{24 \left( \frac{(L-b-c)^2 y(x+w)}{xL} \right)} - V_h \left( \frac{y(x+w)(L-b-c)}{2xL} \right)$$

$$r_2 = \frac{\left\{ M_{hn1} \left( \frac{H(x+w)}{xL} \right) + M_{hp} \left[ \frac{(H-a+y)(x+w)}{xL} + \frac{y}{L-b-c} \right] - M_{vp} \left( \frac{x}{H-a} \right) \right.}{24 \left( \frac{(L-b-c)^2 y(x+w)}{xL} \right) + 24 \left( \frac{x(H-a)(x+w)}{L} \right) + 72 \left( \frac{(a-y)(L-b-c)^2(x+w)}{xL} \right)} - V_h \left( \frac{(a-y)(L-b-c)(x+w)}{xL} \right) - V_v \left( \frac{(x-L+b+c)(x+L-b-c)(x+w)}{2xL} \right)$$

$$r_3 = \frac{M_{vn2} \left( \frac{x+w}{H-a} \right) + M_{vp} \left[ \frac{(x+w)^2}{L(H-a)} + \frac{x+w}{H-a} \right] - V_v \left( \frac{(L-w-x)(x+w)}{L} \right)}{24 \left( \frac{x(H-a)(x+w)}{L} \right) + 24 \left( \frac{w(H-a)(x+w)}{L} \right) + 72 \left( \frac{(H-a)(L-w-x)(x+w)}{L} \right)}$$

$$r_4 = \frac{\left\{ M_{hn2} \left( \frac{H(x+w)}{Lw} \right) + M_{hp} \left[ \frac{(H-a+v)(x+w)}{Lw} + \frac{v}{c} \right] - M_{vp} \left( \frac{w}{H-a} \right) \right.}{24 \left( \frac{w(H-a)(x+w)}{L} \right) + 72 \left( \frac{c^2(a-v)(x+w)}{wL} \right) + 24 \left( \frac{c^2 v(x+w)}{Lw} \right)}$$

$$r_5 = \frac{M_{vn1} \left( \frac{c^2(x+w)}{vwL} \right) + M_{vp} \left( \frac{c^2(x+w)}{vwL} \right) - M_{hp} \left( \frac{v}{c} \right) - V_h \left( \frac{v(x+w)c}{2wL} \right)}{24 \left( \frac{c^2 v(x+w)}{wL} \right)}$$

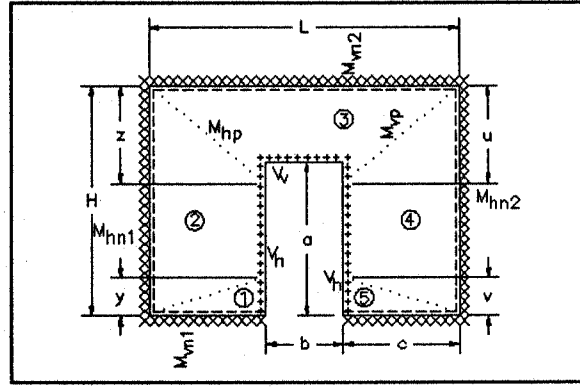
When  $H - \frac{L(H-a)}{x+w} < \frac{y \times L}{(x+w)(L-b-c)}$ , then:

$$\%1 = (L-b-c)(H-a) + xy$$

$$r_1 = \frac{M_{vn1} \left( \frac{(L-b-c)\%1}{Hxy} \right) + M_{vp} \left( \frac{(L-b-c)\%1}{Hxy} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) - V_h \left( \frac{y\%1}{2Hx} \right)}{24 \left( \frac{(L-b-c)y\%1}{Hx} \right)}$$

When  $H - \frac{L(H-a)}{x+w} < \frac{vwL}{c(x+w)}$ , then:

$$r_5 = \frac{\left\{ M_{vn1} \left( \frac{c^2(x+w)}{vwL} \right) + M_{vp} \left( \frac{c^2(x+w)}{vwL} \right) - M_{hp} \left( \frac{v}{c} \right) - V_h \left( \frac{v(x+w)c}{2wL} \right) \right\}}{24 \left( \frac{c^2 v(x+w)}{wL} \right)}$$



Case 8

$$\%1 = u(L-b-c) + cz$$

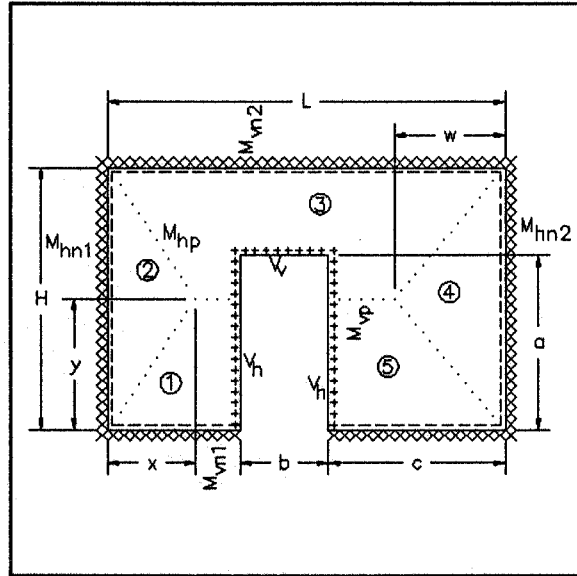
$$r_1 = \frac{M_{vn1} \left( \frac{(L-b-c) \%1}{yLu} \right) + M_{vp} \left( \frac{(L-b-c) \%1}{yLu} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) - V_h \left( \frac{y \%1}{2Lu} \right)}{24 \left( \frac{y(L-b-c) \%1}{Lu} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H \%1}{Lu(L-b-c)} \right) + M_{hp} \left[ \frac{(y+z) \%1}{Lu(y+z)} + \frac{(y+z)}{L-b-c} \right] + V_h \left( \frac{(H-y-z) \%1}{Lu} \right)}{24 \left( \frac{y(L-b-c) \%1}{Lu} \right) + 24 \left( \frac{z(L-b-c) \%1}{Lu} \right) + 72 \left( \frac{(H-y-z)(L-b-c) \%1}{Lu} \right)}$$

$$r_3 = \frac{\left\{ \begin{aligned} &M_{vn2} \left( \frac{\%1}{zu} \right) + M_{vp} \left( \frac{(L-b) \%1}{Lzu} \right) - M_{hp} \left[ \frac{z}{L-b-c} + \frac{u}{c} \right] \\ &- V_h \left[ \frac{(z-H+a)(H-a+z) \%1 + (u-H+a)(u+H-a) \%1}{2Lzu} \right] - V_v \left( \frac{b(H-a) \%1}{Lzu} \right) \end{aligned} \right\}}{24 \left( \frac{z(L-b-c) \%1}{Lu} \right) + 24 \left( \frac{cu \%1}{Lu} \right) + 72 \left( \frac{(H-a)^2 b \%1}{Lzu} \right)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H \%1}{Lcz} \right) + M_{hp} \left[ \frac{(u+v) \%1}{Lcz} + \frac{(u+v)}{c} \right] - V_h \left( \frac{(H-u-v) \%1}{Lz} \right)}{24 \left( \frac{cu \%1}{Lz} \right) + 72 \left( \frac{c(H-u-v) \%1}{Lz} \right) + 24 \left( \frac{vc \%1}{Lz} \right)}$$

$$r_5 = \frac{M_{vn1} \left( \frac{c \%1}{vLz} \right) + M_{vp} \left( \frac{c \%1}{vLz} \right) - M_{hp} \left( \frac{v}{c} \right) - V_h \left( \frac{v \%1}{2Lz} \right)}{24 \left( \frac{vc \%1}{Lz} \right)}$$



Case 9

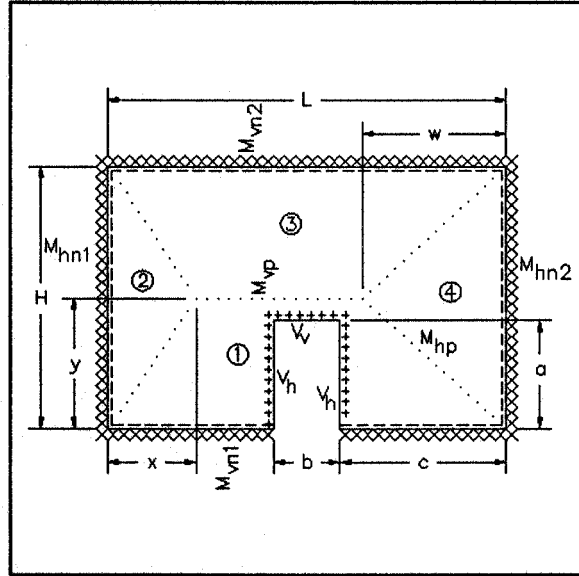
$$r_1 = \frac{M_{vn1} \left( \frac{L-b-c}{y} \right) + M_{vp} \left( \frac{L-b-c}{y} \right) - V_h \left( \frac{y^2}{2} \right)}{24xy + 72y(L-b-c-x)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24xH}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L-b}{H-y} \right) - V_h(a-y)(2H-a-y) - V_v \left( \frac{b(H-a)}{H-y} \right)}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{b(H-a)^2}{H-y} \right) + 24w(H-y)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right)}{24wH}$$

$$r_5 = \frac{M_{vn1} \left( \frac{c}{y} \right) + M_{vp} \left( \frac{c}{y} \right) - V_h \left( \frac{y^2}{2} \right)}{24wy + 72y(c-w)}$$



Case 10

$$r_1 = \frac{M_{vn1} \left( \frac{L-b}{y} \right) + M_{vp} \left( \frac{L}{y} \right) - V_h \left( \frac{a^2}{y} \right) - V_v \left( \frac{b a}{y} \right)}{24 x y + 24 w y + 72 y (L - w - x) - 72 \left( \frac{b a^2}{y} \right)}$$

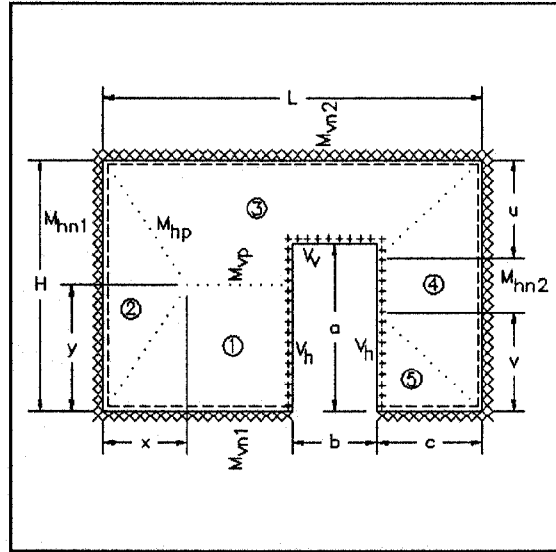
$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24 H x}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L}{H-y} \right)}{24 x (H-y) + 24 w (H-y) + 72 (H-y) (L - w - x)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right)}{24 H w}$$







Case 12

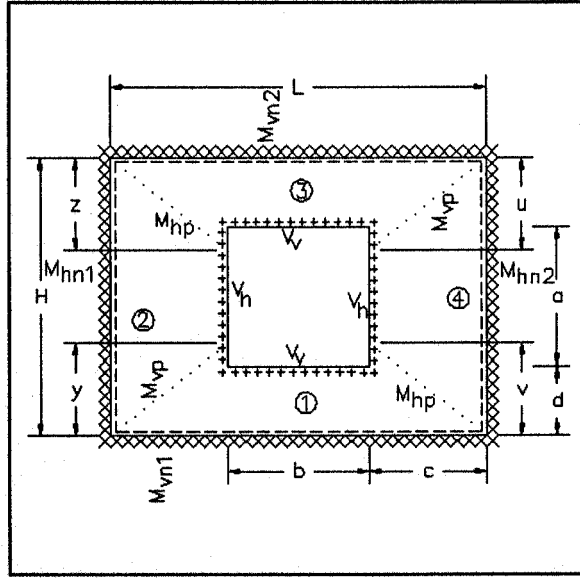
$$r_1 = \frac{M_{vn1} \left( \frac{L-b-c}{y} \right) + M_{vp} \left( \frac{L-b-c}{y} \right) - V_h \left( \frac{y}{2} \right)}{24xy + 72y(L-b-c-x)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L-b}{H-y} \right) - M_{hp} \left( \frac{u}{c} \right) - V_v \left( \frac{b(H-a)}{H-y} \right) - \frac{V_h}{2} \left[ (a-y)(2H-a-y) + \frac{(u-H+a)(u+H-a)}{(H-y)} \right]}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{b(H-a)^2}{(H-y)} \right) + 24 \left( \frac{cu^2}{(H-y)} \right)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{u+v}{c} \right) + M_{hp} \left[ \frac{(u+v)^2}{Hc} + \frac{u+v}{c} \right] - V_h \left( \frac{(H-u-v)(u+v)}{H} \right)}{24 \left( \frac{uc(u+v)}{H} \right) + 72 \left( \frac{(H-u-v)c(u+v)}{H} \right) + 24 \left( \frac{vc(u+v)}{H} \right)}$$

$$r_5 = \frac{M_{vn1} \left( \frac{c}{y} \right) + M_{vp} \left( \frac{c}{y} \right) - M_{hp} \left( \frac{v}{c} \right) - V_h \left( \frac{v^2}{2y} \right)}{24 \left( \frac{cv^2}{y} \right)}$$



Case 13

When  $\frac{H(L-b-c)}{y+z} + \frac{Hc}{u+v} \leq L$ , then:

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L(u+v)}{Hv} \right) + M_{vp} \left( \frac{(L-b)(u+v)}{Hv} \right) - M_{hp} \left[ \frac{y}{L-b-c} + \frac{v}{c} \right] \right.}{24 \left( \frac{(L-b-c)y^2(u+v)}{Hv} \right) + 72 \left( \frac{d^2b(u+v)}{Hv} \right) + 24 \left( \frac{cv(u+v)}{H} \right)}$$

$$\left. - V_h \left[ \left( \frac{(y-d)(y+d)+(v-d)(v+d)}{2} \right) \left( \frac{u+v}{Hv} \right) \right] - V_v \left( \frac{bd(u+v)}{Hv} \right) \right\}$$

$$r_2 = \frac{M_{hn1} \left( \frac{y+z}{L-b-c} \right) + M_{hp} \left[ \frac{(y+z)^2}{H(L-b-c)} + \frac{y+z}{L-b-c} \right] - V_h \left( \frac{(H-y-z)(y+z)}{H} \right)}{24 \left( \frac{y(L-b-c)(y+z)}{H} \right) + 72 \left( \frac{(H-y-z)(L-b-c)(y+z)}{H} \right) + 24 \left( \frac{z(L-b-c)(y+z)}{H} \right)}$$

$$r_3 = \frac{\left\{ M_{vn2} \left( \frac{L(u+v)}{Hu} \right) + M_{vp} \left( \frac{(L-b)(u+v)}{Hu} \right) - M_{hp} \left[ \frac{z}{L-b-c} + \frac{u}{c} \right] - V_v \left( \frac{b(H-a-d)(u+v)}{Hu} \right) \right.}{24 \left( \frac{(L-b-c)z^2(u+v)}{Hu} \right) + 72 \left( \frac{b(H-a-d)^2(u+v)}{Hu} \right) + 24 \left( \frac{cu(u+v)}{H} \right)}$$

$$\left. - V_h \left[ \left( \frac{(z-H+a+d)(z+H-a-d)(u+v)}{2Hu} \right) + \left( \frac{(u-H+a+d)(u+H-a-d)(u+v)}{2Hu} \right) \right] \right\}$$

$$r_4 = \frac{M_{hm2} \left( \frac{u+v}{c} \right) + M_{hp} \left[ \frac{(u+v)^2}{Hc} + \frac{u+v}{c} \right] - V_h \left( \frac{(H-u-v)(u+v)}{H} \right)}{24 \left( \frac{cu(u+v)}{H} \right) + 72 \left( \frac{(H-u-v)c(u+v)}{H} \right) + 24 \left( \frac{cv(u+v)}{H} \right)}$$

When  $\frac{H(L-b-c)}{y+z} + \frac{Hc}{u+v} > L$ , then:

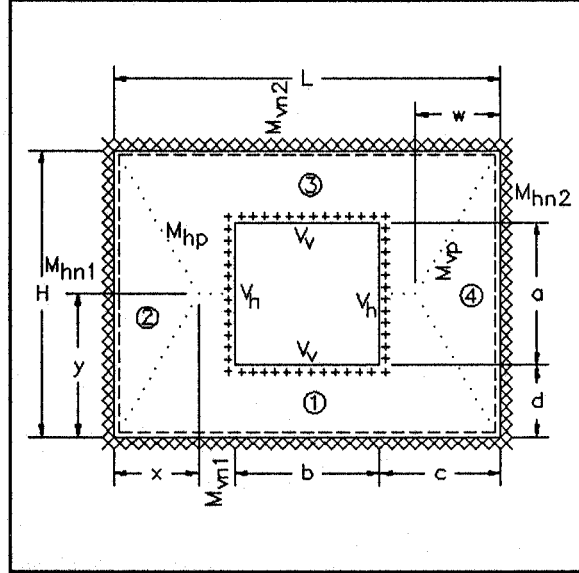
$$\%1 = u(L-b-c) + cz$$

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L \%1}{Luy} \right) + M_{vp} \left( \frac{(L-b) \%1}{Luy} \right) - M_{hp} \left[ \frac{y}{L-b-c} + \frac{v}{c} \right] - V_v \left( \frac{bd \%1}{Luy} \right) \right.}{\left. - V_h \left[ \left( \frac{(y-d)(y+d) \%1}{2Luy} \right) + \left( \frac{(v-d)(v+d) \%1}{2Luy} \right) \right] \right\}}{24 \left( \frac{y(L-b-c) \%1}{Lu} \right) + 72 \left( \frac{d^2 b \%1}{Luy} \right) + 24 \left( \frac{cv^2 \%1}{Luy} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H \%1}{Lu(L-b-c)} \right) + M_{hp} \left[ \frac{(y+z) \%1}{Lu(L-b-c)} + \frac{y+z}{L-b-c} \right] - V_h \left( \frac{(H-y-z) \%1}{Lu} \right)}{24 \left( \frac{y(L-b-c) \%1}{Lu} \right) + 72 \left( \frac{(H-y-z)(L-b-c) \%1}{Lu} \right) + 24 \left( \frac{z(L-b-c) \%1}{Lu} \right)}$$

$$r_3 = \frac{\left\{ M_{vn2} \left( \frac{L \%1}{Luz} \right) + M_{vp} \left( \frac{(L-b) \%1}{Luz} \right) - M_{hp} \left[ \frac{z}{L-b-c} + \frac{u}{c} \right] - V_v \left( \frac{b(H-a-d) \%1}{Luz} \right) \right.}{\left. - V_h \left[ \left( \frac{(z-H+a+d)(z+H-a-d) \%1}{2Luz} \right) + \left( \frac{(u-H+a+d)(u+H-a-d) \%1}{2Luz} \right) \right] \right\}}{24 \left( \frac{z(L-b-c) \%1}{Lu} \right) + 72 \left( \frac{b(H-a-d)^2 \%1}{Luz} \right) + 24 \left( \frac{cu \%1}{Lz} \right)}$$

$$r_4 = \frac{M_{hm2} \left( \frac{H \%1}{Lcz} \right) + M_{hp} \left[ \frac{(u+v) \%1}{Lcz} + \frac{u+v}{c} \right] - V_h \left( \frac{(H-u-v) \%1}{Lz} \right)}{24 \left( \frac{cu \%1}{Lz} \right) + 72 \left( \frac{c(H-u-v) \%1}{Lz} \right) + 24 \left( \frac{vc \%1}{Lz} \right)}$$



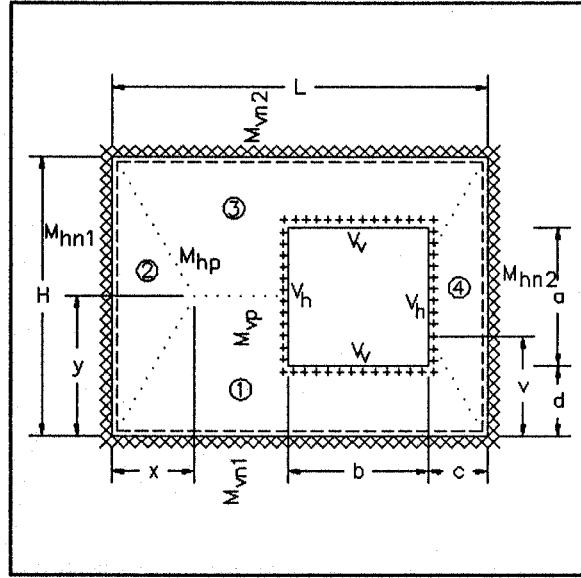
Case 14

$$r_1 = \frac{M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left( \frac{L-b}{y} \right) - V_h \left( \frac{(y-d)(y+d)}{y} \right) - V_v \left( \frac{bd}{y} \right)}{24xy + 72y(L-b-c-x) + 72 \left( \frac{bd^2}{y} \right) + 72(c-w)y + 24wy}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L-b}{H-y} \right) - V_h \left( \frac{(a+d-y)(a+d+y)}{H-y} \right) - V_v \left( \frac{b(H-a)}{H-y} \right)}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{b(H-a-d)^2}{H-y} \right) + 72(c-w)(H-y) + 24w(H-y)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right)}{24Hw}$$



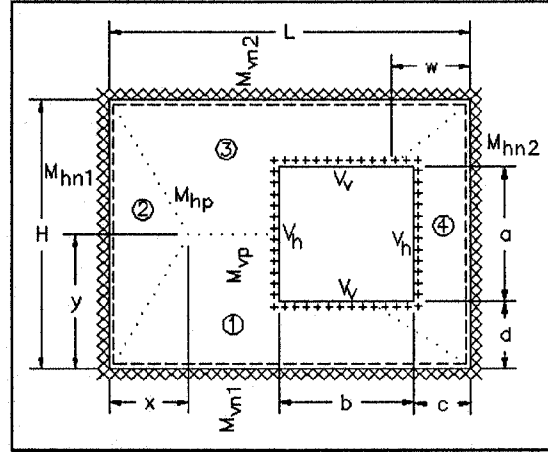
Case 15

$$r_1 = \frac{M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left( \frac{L-b}{y} \right) - M_{hp} \left( \frac{v}{c} \right) - V_h \left[ \frac{(y-d)(y+d) + (v-d)(v+d)}{2y} \right] - V_v \left( \frac{bd}{y} \right)}{24xy + 72y(L-b-c-x) + 72 \left( \frac{bd^2}{y} \right) + 24 \left( \frac{cv^2}{y} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{\left\{ \begin{aligned} &M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L-b}{H-y} \right) - M_{hp} \left( \frac{v(H-y)}{yc} \right) - V_v \left( \frac{b(H-a-d)}{(H-y)} \right) \\ &- V_h \left[ \frac{(a+d-y)(2H-a-d-y)}{2(H-y)} \right] + \frac{\left( \frac{v(H-y)}{y} - H+a+d \right) \left( \frac{v(H-y)}{y} + H-a-d \right)}{2(H-y)} \end{aligned} \right\}}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{b(H-a-d)^2}{H-y} \right) + 24 \left( \frac{cv^2(H-y)}{y^2} \right)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{Hv}{cy} \right) + M_{hp} \left[ \frac{v^2}{cy} + \frac{v^2(H-y)}{cy^2} + \frac{v(H-y)}{cy} + \frac{v}{c} \right] - V_h \left[ \frac{Hv}{y} - \frac{v^2}{y} - \frac{v^2(H-y)}{y^2} \right]}{24 \left( \frac{cv^2(H-y)}{y^2} \right) + 72 \left[ \frac{Hcv}{y} - \frac{cv^2}{y} - \frac{cv^2(H-y)}{y^2} \right] + 24 \left( \frac{cv^2}{y} \right)}$$



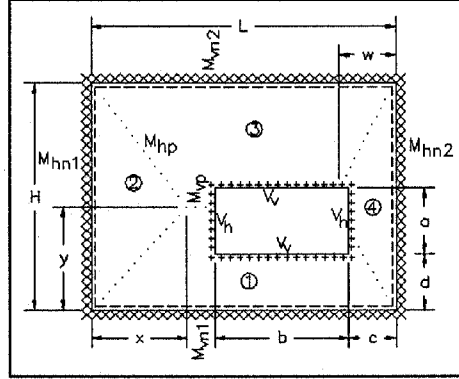
Case 16

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left[ \frac{L-b-c}{y} + \frac{w(H-y)d}{(H-a-d)y^2} + \frac{w(H-y)}{(H-a-d)y} \right] \right.}{24xy + 72y(L-b-c-x) + 72 \left[ \left( b+c - \frac{w(H-y)d}{(H-a-d)y} \right) \left( \frac{d^2}{y} \right) + 24 \left( \frac{w(H-y)d^3}{(H-a-d)y^2} \right) \right.}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left[ \frac{L-b-c-w}{H-y} + \frac{w}{H-a-d} \right] - V_h \left( \frac{(a+d-y)(2H-a-d-y)}{2(H-y)} \right) - V_v \left( \frac{(b+c-w)(H-a-d)}{H-y} \right)}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{(b+c-w)(H-a-d)^2}{H-y} \right) + 24 \left( \frac{w(H-a-d)^2}{H-y} \right)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H(H-a-d)}{w(H-y)} \right) + M_{hp} \left( \frac{(H-a)(H-a-d)}{w(H-y)} \right) - M_{vp} \left[ \frac{w}{H-a-d} + \frac{w(H-y)d}{(H-a-d)cy} \right] - V_h \left( \frac{ac(H-a-d)}{w(H-y)} \right)}{24 \left( \frac{w(H-a-d)^2}{H-y} \right) + 72 \left( \frac{ac^2(H-a-d)}{w(H-y)} \right) + 24 \left( \frac{w(H-y)d^3}{(H-a-d)y^2} \right)}$$



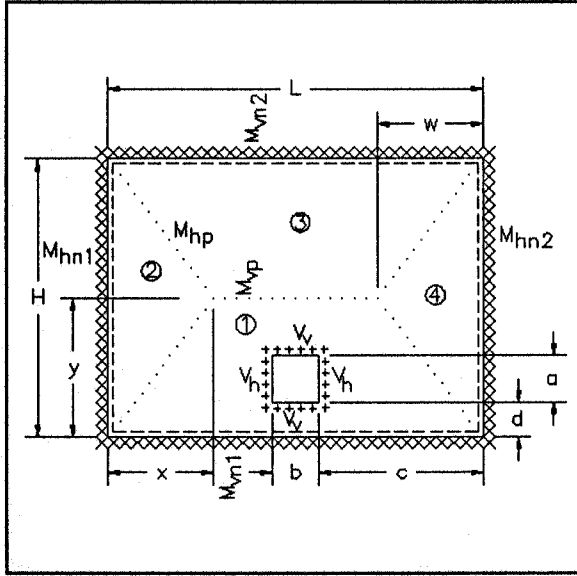
Case 17

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left( \frac{L-b}{y} \right) - M_{hp} \left( \frac{y(H-a-d)}{w(H-y)} \right) - V_v \left( \frac{bd}{y} \right) \right\}}{24xy + 72y(L-b-c-x) + 72 \left( \frac{bd^2}{y} \right) + 24 \left( \frac{c^3y(H-a-d)^2}{w^2(H-y)^2} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left[ \frac{L-b-c+w}{H-y} + \frac{w}{H-a-d} \right] - V_h \left[ \frac{(a+d-y)(2H-a-d-y)}{2(H-y)} \right] - V_v \left( \frac{(b+c-w)(H-a-d)}{H-y} \right)}{24x(H-y) + 72(L-b-c-x)(H-y) + 72 \left( \frac{(b+c-w)(H-a-d)^2}{H-y} \right) + 24 \left( \frac{w(H-a-d)^2}{H-y} \right)}$$

$$r_4 = \frac{\left\{ M_{hn2} \left( \frac{H(H-a-d)}{w(H-y)} \right) + M_{hp} \left[ \left( H-a + \frac{cy(H-a-d)}{w(H-y)} \right) \left( \frac{H-a-d}{w(H-y)} \right) + \frac{y(H-a-d)}{w(H-y)} \right] \right.}{24 \left( \frac{w(H-a-d)^2}{H-y} \right) + 72 \left[ \left( a - \frac{cy(H-a-d)}{w(H-y)} \right) \left( \frac{c^2(H-a-d)}{w(H-y)} \right) \right] + 24 \left( \frac{c^3y(H-a-d)^2}{w^2(H-y)^2} \right)}$$



Case 18

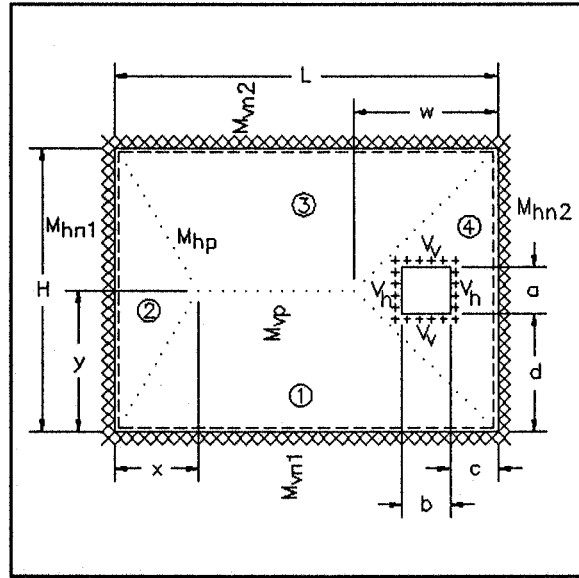
$$r_1 = \frac{M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left( \frac{L}{y} \right) - V_h \left( \frac{a(a+2d)}{y} \right) - V_v \left( \frac{b(a+2d)}{y} \right)}{24xy + 72y(L-x-w) + 24wy - 72 \left( \frac{ab(a+2d)}{y} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24Hx}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H-y} \right) + M_{vp} \left( \frac{L}{H-y} \right)}{24x(H-y) + 72(L-x-w)(H-y) + 24w(H-y)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right)}{24Hw}$$





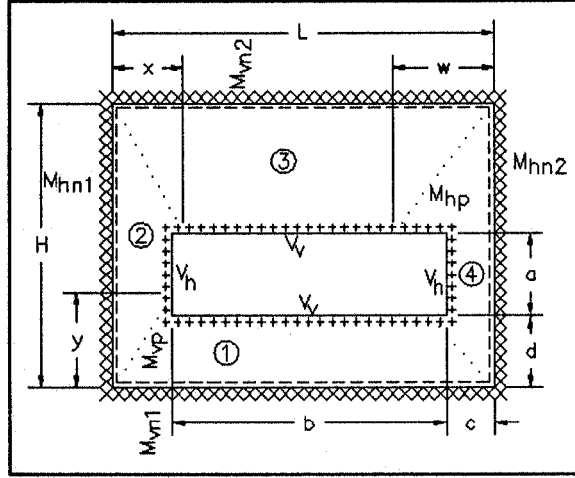
Case 19

$$r_1 = \frac{M_{vn1} \left( \frac{L}{y} \right) + M_{vp} \left( \frac{L}{y} \right)}{24 x y + 72 y (L - x - w) + 24 w y}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H}{x} \right) + M_{hp} \left( \frac{H}{x} \right)}{24 H x}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L}{H - y} \right) + M_{vp} \left( \frac{L}{H - y} \right)}{24 x (H - y) + 72 (L - x - w) (H - y) + 24 w (H - y)}$$

$$r_4 = \frac{M_{hn2} \left( \frac{H}{w} \right) + M_{hp} \left( \frac{H}{w} \right) - V_h \left( \frac{a(b + 2c)}{w} \right) - V_v \left( \frac{b(b + 2c)}{w} \right)}{24 H w - 72 \left( \frac{a b (b + 2c)}{w} \right)}$$



Case 20

When  $\frac{H(L-b-c)(x+w)}{(H-a-d)(L-b-c) + xy} \leq L$ , then:

$$\%1 = (H-a-d)(L-b-c) + xy$$

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L\%1}{Hxy} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) + M_{vp} \left[ \left( L-b-c + \frac{w(L-b-c)d}{xy} \right) \left( \frac{\%1}{Hxy} \right) + \frac{w(L-b-c)}{xy} \right] \right.}{24 \left( \frac{(L-b-c)y\%1}{Hx} \right) + 72 \left[ \left( b+c - \frac{w(L-b-c)d}{xy} \right) \left( \frac{d^2\%1}{Hxy} \right) \right] + 24 \left( \frac{w(L-b-c)d^3\%1}{x^2y^2H} \right)}$$

$$r_2 = \frac{\left\{ M_{hn1} \left( \frac{\%1}{x(L-b-c)} \right) + M_{hp} \left[ \frac{(H-a-d+y)\%1}{Hx(L-b-c)} + \frac{y}{L-b-c} \right] - M_{vp} \left( \frac{x}{H-a-d} \right) \right.}{24 \left( \frac{(L-b-c)\%1}{Hx} \right) + 72 \left( \frac{(a+d-y)(L-b-c)\%1}{Hx} \right) + 24 \left( \frac{x(H-a-d)\%1}{H(L-b-c)} \right)}$$

$$r_3 = \frac{M_{vn2} \left( \frac{L\%1}{H(H-a-d)(L-b-c)} \right) + M_{vp} \left[ \frac{(x+w)\%1}{H(H-a-d)(L-b-c)} + \frac{x+w}{H-a-d} \right] - V_v \left( \frac{(L-x-w)\%1}{H(L-b-c)} \right)}{24 \left( \frac{(H-a-d)x\%1}{H(L-b-c)} \right) + 72 \left( \frac{(L-w-x)(H-a-d)\%1}{H(L-b-c)} \right) + 24 \left( \frac{w(H-a-d)\%1}{H(L-b-c)} \right)}$$

$$r_4 = \frac{\left\{ \begin{aligned} &M_{hm2} \left( \frac{H(H-a-d)(y+z)}{Hwz} \right) + M_{hp} \left( \frac{(H-a)(H-a-d)(y+z)}{Hwz} \right) - M_{vp} \left[ \frac{w}{H-a-d} + \frac{wz}{y(H-a-d)} \right] \\ &- V_h \left( \frac{ac(H-a-d)(y+z)}{Hwz} \right) - V_v \left[ (w-c)(w+c) + \left( \frac{wzd}{(H-a-d)y} - c \right) \left( \frac{wzd}{(H-a-d)y} + c \right) \right] \left[ \frac{(H-a-d)(y+z)}{2Hwz} \right] \end{aligned} \right\}}{24 \left( \frac{w(H-a-d)^2(y+z)}{Hz} \right) + 72 \left( \frac{ac^2(H-a-d)(y+z)}{Hwz} \right) + 24 \left( \frac{wzd^3(y+z)}{(H-a-d)y^2H} \right)}$$

When  $\frac{H(L-b-c)}{y+z} + \frac{Hwz}{(H-a-d)(y+z)} > L$ , then:

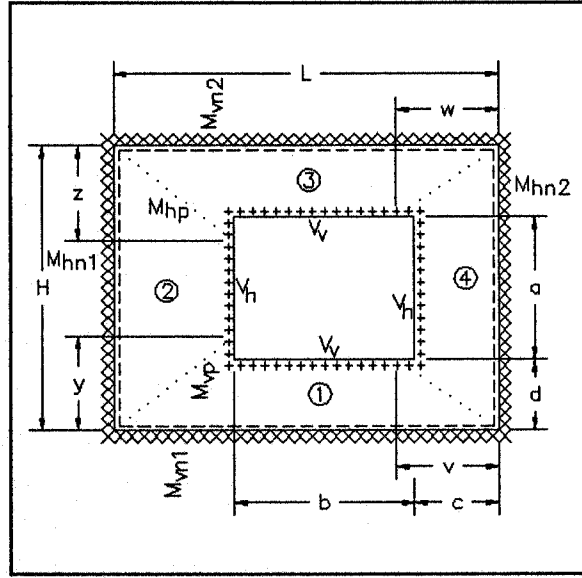
$$\%1 = (L-b-c)(H-a-d) + wz$$

$$r_1 = \frac{\left\{ \begin{aligned} &M_{vn1} \left( \frac{\%1}{y(H-a-d)} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) + M_{vp} \left[ \left( L-b-c + \frac{wzd}{y(H-a-d)} \right) \left( \frac{\%1}{L(H-a-d)y} \right) + \frac{wz}{(H-a-d)y} \right] \\ &- V_h \left( \frac{(y-d)(y+d)\%1}{2L(H-a-d)y} \right) - V_v \left[ \left( b+c - \frac{wzd}{(H-a-d)y} \right) \left( \frac{d\%1}{L(H-a-d)y} \right) \right] \end{aligned} \right\}}{24 \left( \frac{(L-b-c)y\%1}{L(H-a-d)} \right) + 72 \left[ \left( b+c - \frac{wzd}{(H-a-d)y} \right) \left( \frac{d^2\%1}{L(H-a-d)y} \right) \right] + 24 \left( \frac{wzd^3\%1}{(H-a-d)y^2L} \right)}$$

$$r_2 = \frac{M_{hm1} \left( \frac{H\%1}{L(L-b-c)(H-a-d)} \right) + M_{hp} \left[ \frac{(y+z)\%1}{L(L-b-c)(H-a-d)} + \frac{y+z}{L-b-c} \right] - V_h \left( \frac{(H-y-z)\%1}{L(H-a-d)} \right)}{24 \left( \frac{(L-b-c)y\%1}{L(H-a-d)} \right) + 72 \left( \frac{(H-y-z)(L-b-c)\%1}{L(H-a-d)} \right) + 24 \left( \frac{z(L-b-c)\%1}{L(H-a-d)} \right)}$$

$$r_3 = \frac{\left\{ \begin{aligned} &M_{vn2} \left( \frac{\%1}{z(H-a-d)} \right) + M_{vp} \left[ \frac{(L-b-c+w)\%1}{L(H-a-d)z} + \frac{w}{H-a-d} \right] - M_{hp} \left( \frac{z}{L-b-c} \right) \\ &- V_h \left( \frac{(z-H+a+d)(z+H-a-d)\%1}{2L(H-a-d)z} \right) - V_v \left( \frac{(b+c-w)\%1}{Lz} \right) \end{aligned} \right\}}{24 \left( \frac{z(L-b-c)\%1}{L(H-a-d)} \right) + 72 \left( \frac{(b+c-w)(H-a-d)\%1}{Lz} \right) + 24 \left( \frac{w(H-a-d)\%1}{Lz} \right)}$$

$$r_4 = \frac{\left\{ \begin{aligned} &M_{hm2} \left( \frac{H\%1}{Lwz} \right) + M_{hp} \left( \frac{(H-a)\%1}{Lwz} \right) - M_{vp} \left[ \frac{w}{H-a-d} + \frac{wz}{y(H-a-d)} \right] - V_h \left( \frac{ac\%1}{Lwz} \right) \\ &- V_v \left[ \left( \frac{(w-c)(w+c)\%1}{2Lwz} \right) + \left( \frac{wzd}{(H-a-d)y} - c \right) \left( \frac{wzd}{(H-a-d)y} + c \right) \right] \left( \frac{\%1}{2Lwz} \right) \end{aligned} \right\}}{24 \left( \frac{w(H-a-d)\%1}{Lz} \right) + 72 \left( \frac{ac^2\%1}{Lwz} \right) + 24 \left( \frac{wzd^3\%1}{(H-a-d)^2y^2L} \right)}$$



Case 21

When  $\frac{H(L-b-c)}{y+z} + \frac{Hwz}{(H-a-d)(y+z)} \leq L$ , then:

$$r_1 = \frac{\left\{ M_{vn1} \left( \frac{L(y+z)}{Hy} \right) + M_{vp} \left[ \left( L-b-c + \frac{wzd}{y(H-a-d)} \right) \left( \frac{y+z}{Hy} \right) + \frac{wz}{(H-a-d)y} \right] - M_{hp} \left( \frac{y}{L-b-c} \right) \right.}{24 \left( \frac{(L-b-c)y(y+z)}{H} \right) + 72 \left[ \left( b+c - \frac{wzd}{(H-a-d)y} \right) \left( \frac{d^2(y+z)}{Hy} \right) \right] + 24 \left( \frac{wzd^3(y+z)}{(H-a-d)y^2H} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{y+z}{L-b-c} \right) + M_{hp} \left[ \frac{(y+z)^2}{H(L-b-c)} + \frac{y+z}{L-b-c} \right] - V_h \left( \frac{(H-y-z)(y+z)}{H} \right)}{24 \left( \frac{(L-b-c)y(y+z)}{H} \right) + 72 \left( \frac{(H-y-z)(L-b-c)(y+z)}{H} \right) + 24 \left( \frac{z(L-b-c)(y+z)}{H} \right)}$$

$$r_3 = \frac{\left\{ M_{vn2} \left( \frac{L(y+z)}{Hz} \right) + M_{vp} \left[ \frac{(L-b-c+w)(y+z)}{Hz} + \frac{w}{H-a-d} \right] - M_{hp} \left( \frac{z}{L-b-c} \right) \right.}{24 \left( \frac{z(L-b-c)(y+z)}{H} \right) + 72 \left( \frac{(b+c-w)(H-a-d)^2(y+z)}{Hz} \right) + 24 \left( \frac{w(H-a-d)^2(y+z)}{Hz} \right)}$$

$$r_4 = \frac{\left\{ \begin{aligned} &M_{hn2} \left( \frac{H(H-a-d)(y+z)}{Hwz} \right) + M_{hp} \left( \frac{(H-a)(H-a-d)(y+z)}{Hwz} \right) - M_{vp} \left[ \frac{w}{H-a-d} + \frac{wz}{y(H-a-d)} \right] \\ &- V_h \left( \frac{ac(H-a-d)(y+z)}{Hwz} \right) - V_v \left[ (w-c)(w+c) + \left( \frac{wzd}{(H-a-d)y} - c \right) \left( \frac{wzd}{(H-a-d)y} + c \right) \right] \left[ \frac{(H-a-d)(y+z)}{2Hwz} \right] \end{aligned} \right\}}{24 \left( \frac{w(H-a-d)^2(y+z)}{Hz} \right) + 72 \left( \frac{ac^2(H-a-d)(y+z)}{Hwz} \right) + 24 \left( \frac{wzd^3(y+z)}{(H-a-d)y^2H} \right)}$$

When  $\frac{H(L-b-c)}{y+z} + \frac{Hwz}{(H-a-d)(y+z)} > L$ , then:

$$\%1 = (L-b-c)(H-a-d) + wz$$

$$r_1 = \frac{\left\{ \begin{aligned} &M_{vn1} \left( \frac{\%1}{y(H-a-d)} \right) - M_{hp} \left( \frac{y}{L-b-c} \right) + M_{vp} \left[ \left( L-b-c + \frac{wzd}{y(H-a-d)} \right) \left( \frac{\%1}{L(H-a-d)y} \right) + \frac{wz}{(H-a-d)y} \right] \\ &- V_h \left( \frac{(y-d)(y+d)\%1}{2L(H-a-d)y} \right) - V_v \left[ \left( b+c - \frac{wzd}{(H-a-d)y} \right) \left( \frac{d\%1}{L(H-a-d)y} \right) \right] \end{aligned} \right\}}{24 \left( \frac{(L-b-c)y\%1}{L(H-a-d)} \right) + 72 \left[ \left( b+c - \frac{wzd}{(H-a-d)y} \right) \left( \frac{d^2\%1}{L(H-a-d)y} \right) \right] + 24 \left( \frac{wzd^3\%1}{(H-a-d)^2y^2L} \right)}$$

$$r_2 = \frac{M_{hn1} \left( \frac{H\%1}{L(L-b-c)(H-a-d)} \right) + M_{hp} \left[ \frac{(y+z)\%1}{L(L-b-c)(H-a-d)} + \frac{y+z}{L-b-c} \right] - V_h \left( \frac{(H-y-z)\%1}{L(H-a-d)} \right)}{24 \left( \frac{(L-b-c)y\%1}{L(H-a-d)} \right) + 72 \left( \frac{(H-y-z)(L-b-c)\%1}{L(H-a-d)} \right) + 24 \left( \frac{z(L-b-c)\%1}{L(H-a-d)} \right)}$$

$$r_3 = \frac{\left\{ \begin{aligned} &M_{vn2} \left( \frac{\%1}{z(H-a-d)} \right) + M_{vp} \left[ \frac{(L-b-c+w)\%1}{L(H-a-d)z} + \frac{w}{H-a-d} \right] - M_{hp} \left( \frac{z}{L-b-c} \right) \\ &- V_h \left( \frac{(z-H+a+d)(z+H-a-d)\%1}{2L(H-a-d)z} \right) - V_v \left( \frac{(b+c-w)\%1}{Lz} \right) \end{aligned} \right\}}{24 \left( \frac{z(L-b-c)\%1}{L(H-a-d)} \right) + 72 \left( \frac{(b+c-w)(H-a-d)\%1}{Lz} \right) + 24 \left( \frac{w(H-a-d)\%1}{Lz} \right)}$$

$$r_4 = \frac{\left\{ \begin{aligned} &M_{hn2} \left( \frac{H\%1}{Lwz} \right) + M_{hp} \left( \frac{(H-a)\%1}{Lwz} \right) - M_{vp} \left[ \frac{w}{H-a-d} + \frac{wz}{y(H-a-d)} \right] - V_h \left( \frac{ac\%1}{Lwz} \right) \\ &- V_v \left[ \left( \frac{(w-c)(w+c)\%1}{2Lwz} \right) + \left( \frac{wzd}{(H-a-d)y} - c \right) \left( \frac{wzd}{(H-a-d)y} + c \right) \left( \frac{\%1}{2Lwz} \right) \right] \end{aligned} \right\}}{24 \left( \frac{w(H-a-d)\%1}{Lz} \right) + 72 \left( \frac{ac^2\%1}{Lwz} \right) + 24 \left( \frac{wzd^3\%1}{(H-a-d)^2y^2L} \right)}$$

**VIEWGRAPHS 1 THROUGH 27**

# **YIELD-LINE ANALYSIS OF SLABS WITH COVERED OPENINGS**

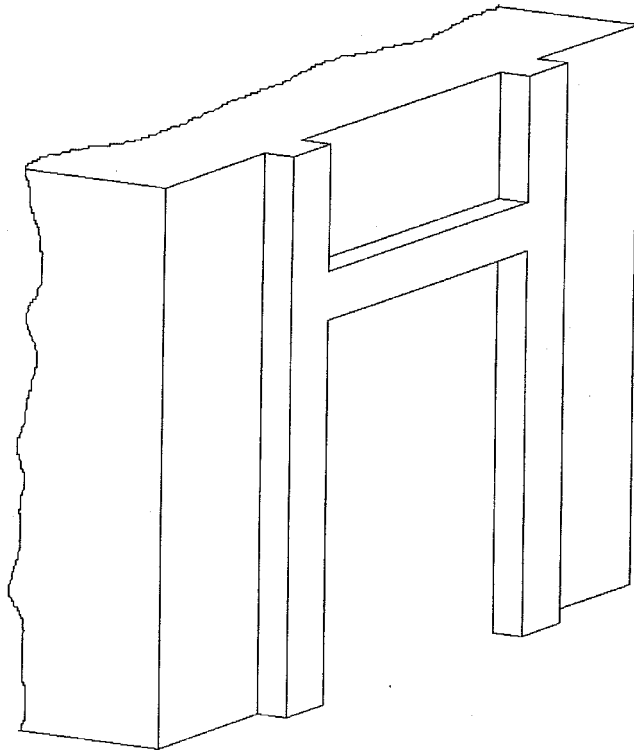
**26th Dept of Defense Explosives Safety Seminar  
Miami, Florida**

**16-18 August 1994**

**by**

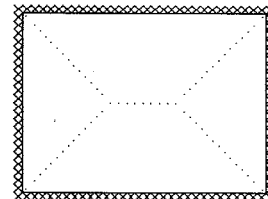
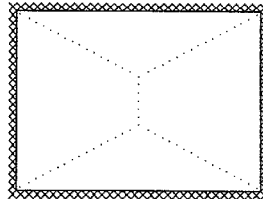
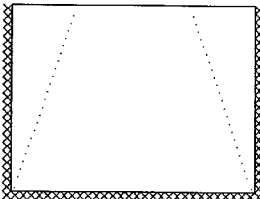
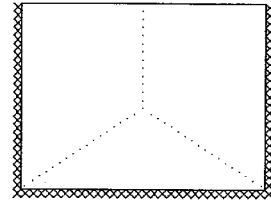
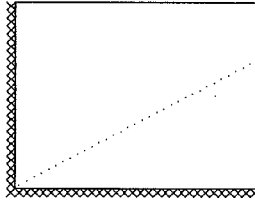
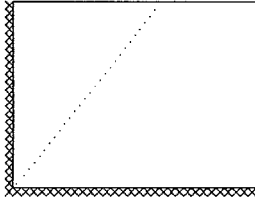
**Phil Wager, MSCE  
Naval Facilities Engineering Service Center  
Port Hueneme, California**

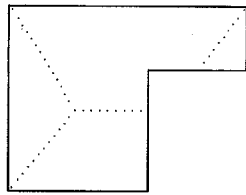




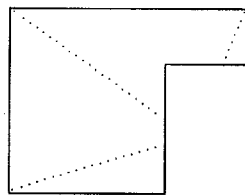


## Cases With Published Criteria

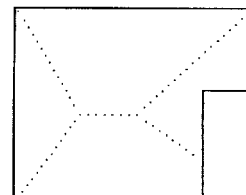




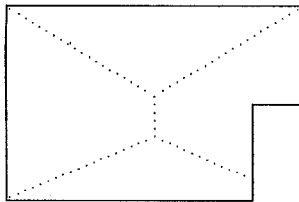
Case 1



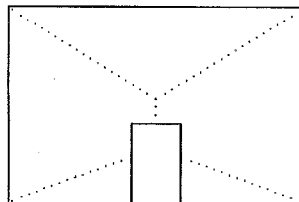
Case 2



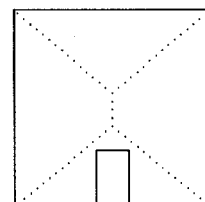
Case 3



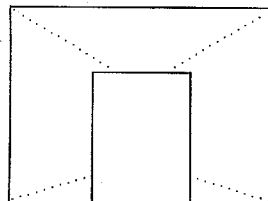
Case 4



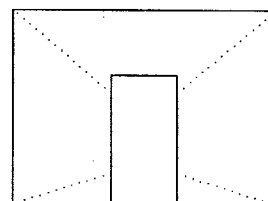
Case 5



Case 6

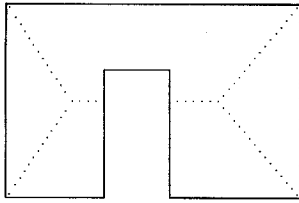


Case 7

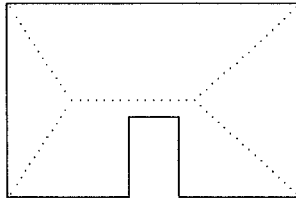


Case 8

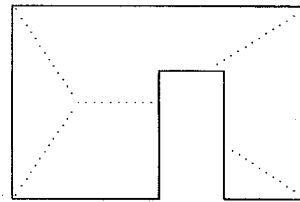




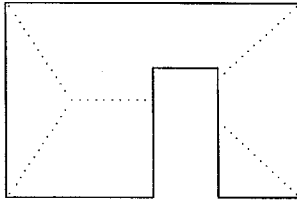
Case 9



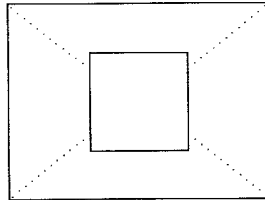
Case 10



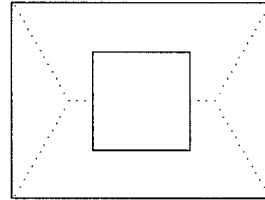
Case 11



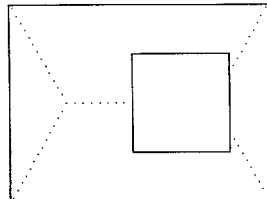
Case 12



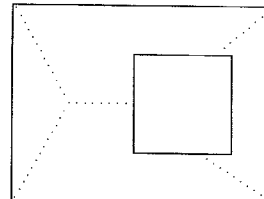
Case 13



Case 14

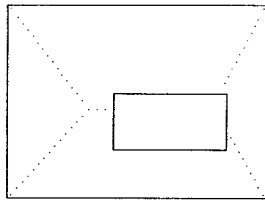


Case 15

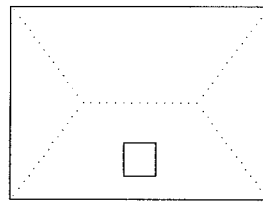


Case 16

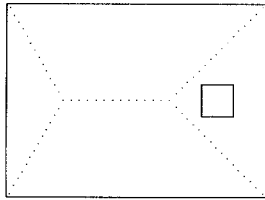




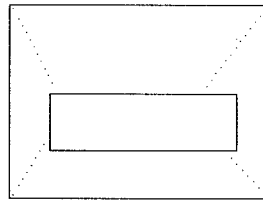
Case 17



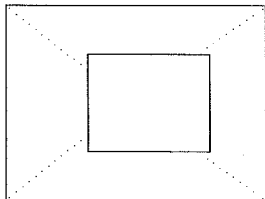
Case 18



Case 19



Case 20

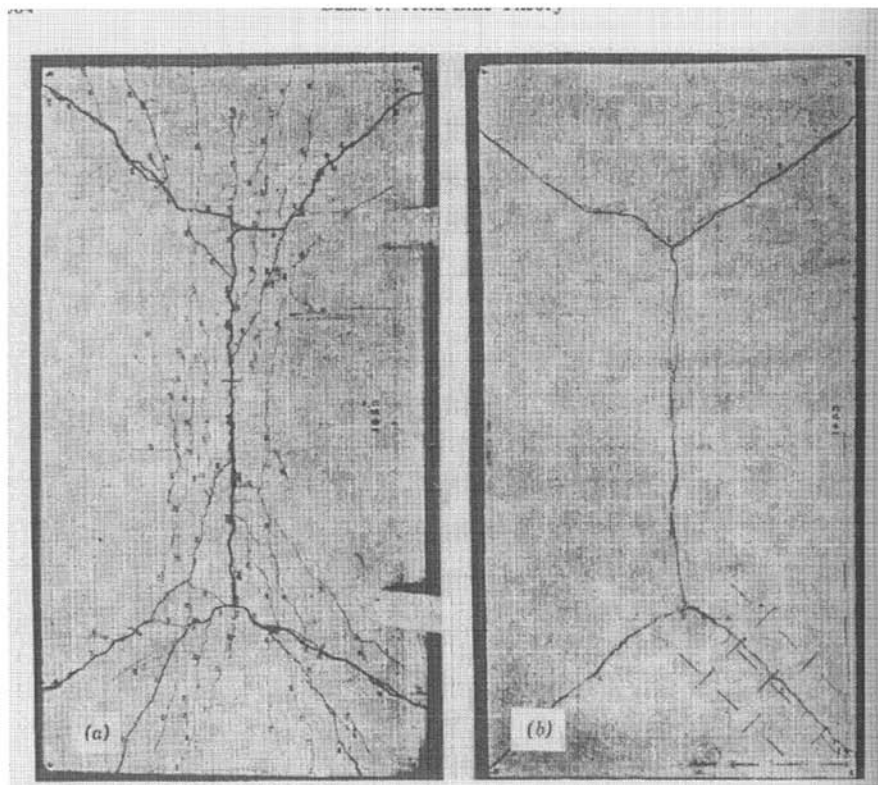


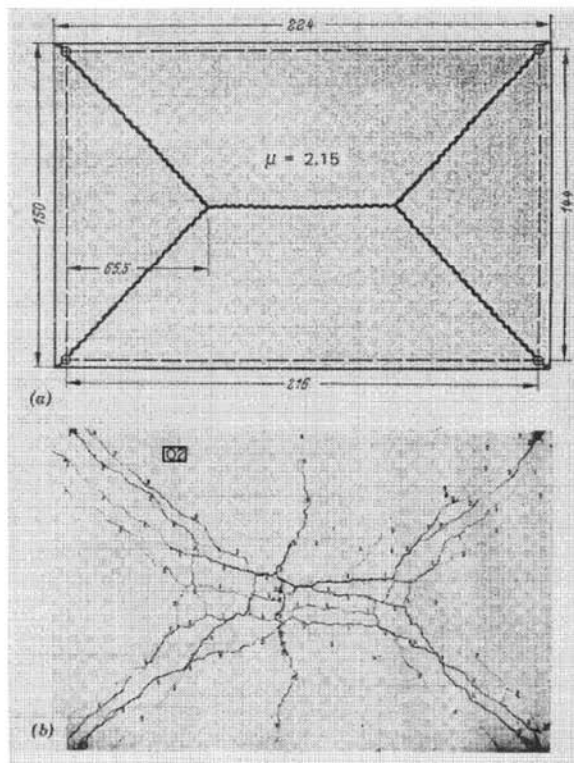
Case 21



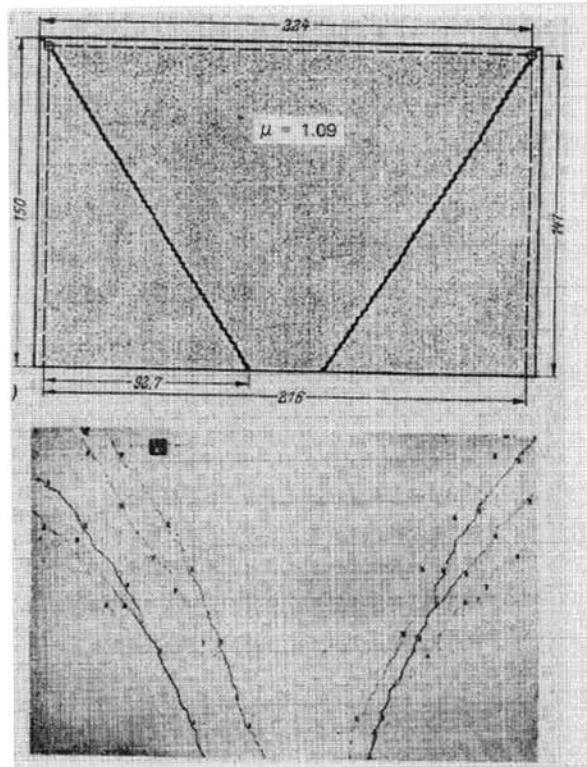


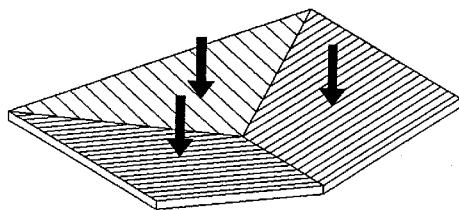






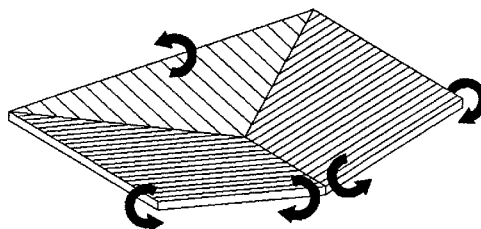






**External Work**

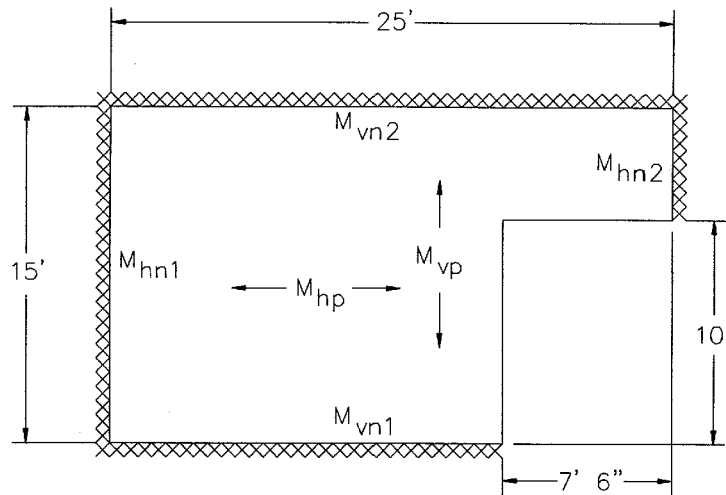
=



**Internal Work**



## Sample Problem



## **Problem Parameters**

**Wall Thickness = 36 in.**

**Steel Reinforcement = #8 bars @ 12 in. o.c.**

**#9 bars @ 12 in. o.c.**

**Allowable support rotation = 8 deg**

**Door = A-36 steel, 2 in. thick**

**Supported along top and two sides**



## **Step 1: Compute the Moment Capacity of Walls**

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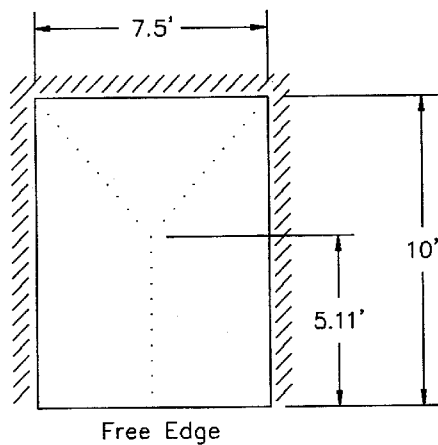
**Use standard computer program  
such as BARCS**

- $M_{hn1} = 187,393 \text{ in-lbs/in}$
- $M_{hn2} = 187,393 \text{ in-lbs/in}$
- $M_{hp} = 187,393 \text{ in-lbs/in}$
- $M_{vn1} = 238,141 \text{ in-lbs/in}$
- $M_{vn2} = 238,141 \text{ in-lbs/in}$
- $M_{vp} = 238,141 \text{ in-lbs/in}$



## Step 2: Compute Ultimate Resistance of Door and Corresponding Line Loads

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From TM 5-1300  
and BARCS:

$$r_u = 72.0 \text{ psi}$$

$$V_h = 29,375 \text{ lbs/ft}$$

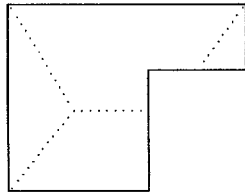
$$V_v = 25,345 \text{ lbs/ft}$$



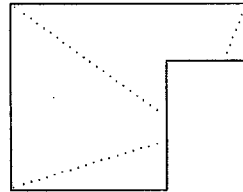
### **Step 3: Compute Ultimate Resistance of Slab**

---

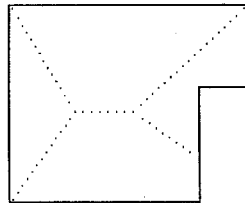
#### **Four Failure Mechanisms Apply**



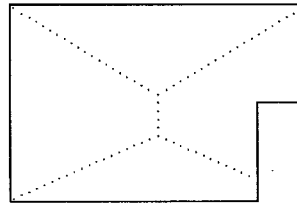
Case 1



Case 2



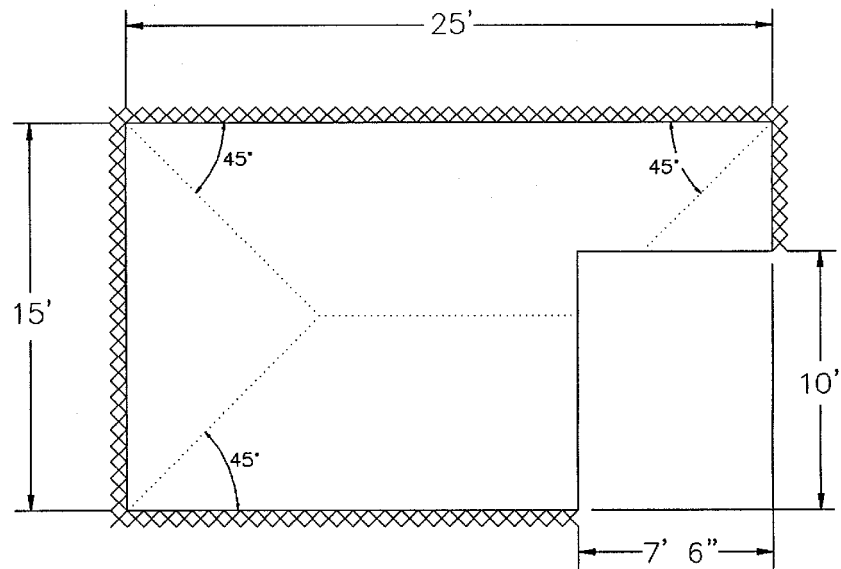
Case 3



Case 4

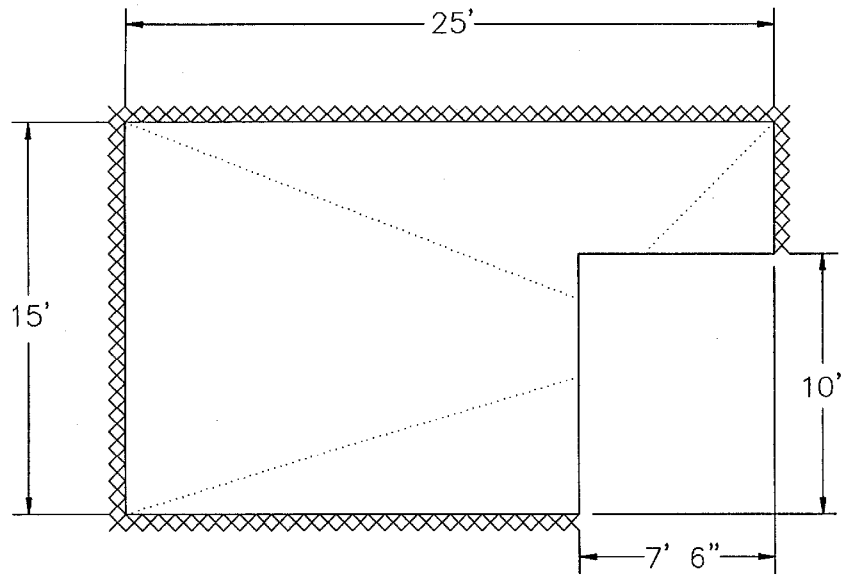


## Approximate Location of Yield Lines

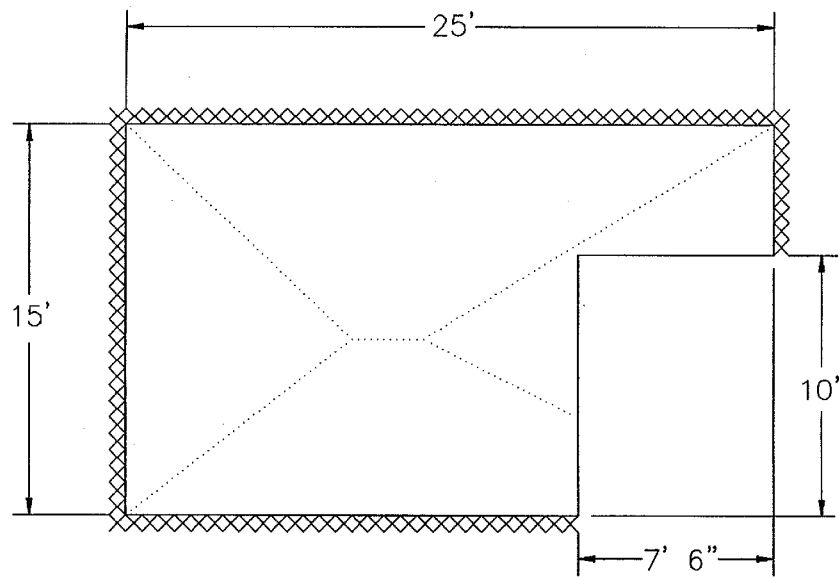




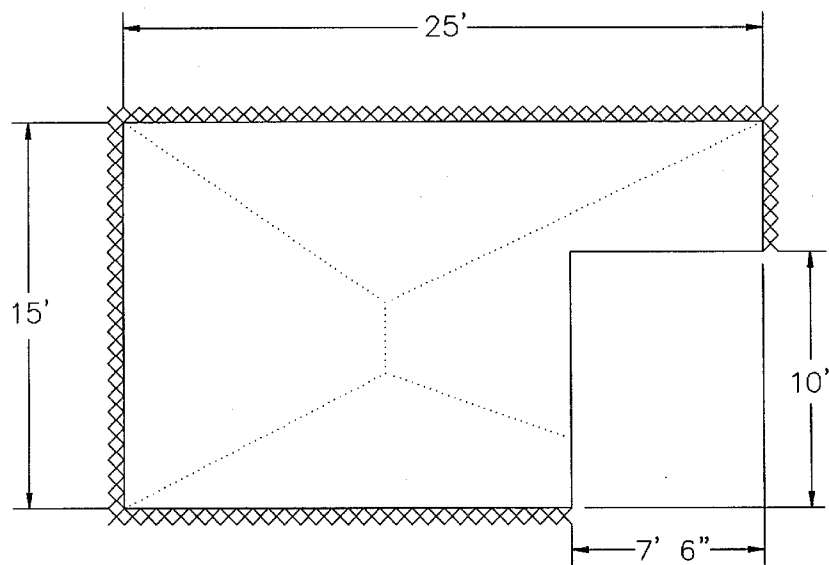
## Approximate Location of Yield Lines -- Case 2



### Approximate Location of Yield Lines -- Case 3



## Approximate Location of Yield Lines -- Case 4



## **Summary of Ultimate Resistance, $r_u$**

---

<b>Case</b>	<b><math>r_u</math></b>
<b>1</b>	<b>189.1 psi</b>
<b>2</b>	<b>did not converge</b>
<b>3</b>	<b>did not converge</b>
<b>4</b>	<b>did not converge</b>



**Step 4: Compute Average Moment Capacity and Stiffness of Equivalent Slab 1**

---

$$\begin{aligned} M_{avg} &= [M_{hn1} + M_{hn2} + M_{hp} + M_{vn1} \\ &\quad + M_{vn2} + M_{vp}] / 6 \\ &= [3(187,393) + 3(238,141)] / 6 \\ &= 213,767 \text{ in.-lb/in.} \end{aligned}$$

**From TM 5-1300; Computer Program BARCS:**

$$K_{e1} = 2053.9 \text{ psi-in.}$$



### Step 5: Model Equivalent Slab 2

$$a/H = 10 \text{ ft} / 15 \text{ ft} = 0.6667$$

$$b/L = 25 \text{ ft} / 25 \text{ ft} = 0.3000$$

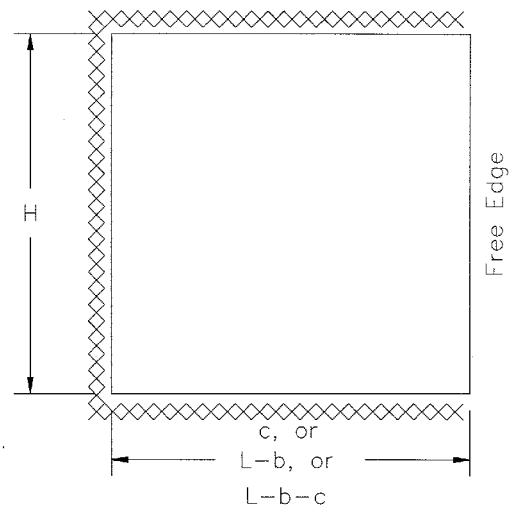
$$a/H > b/L$$

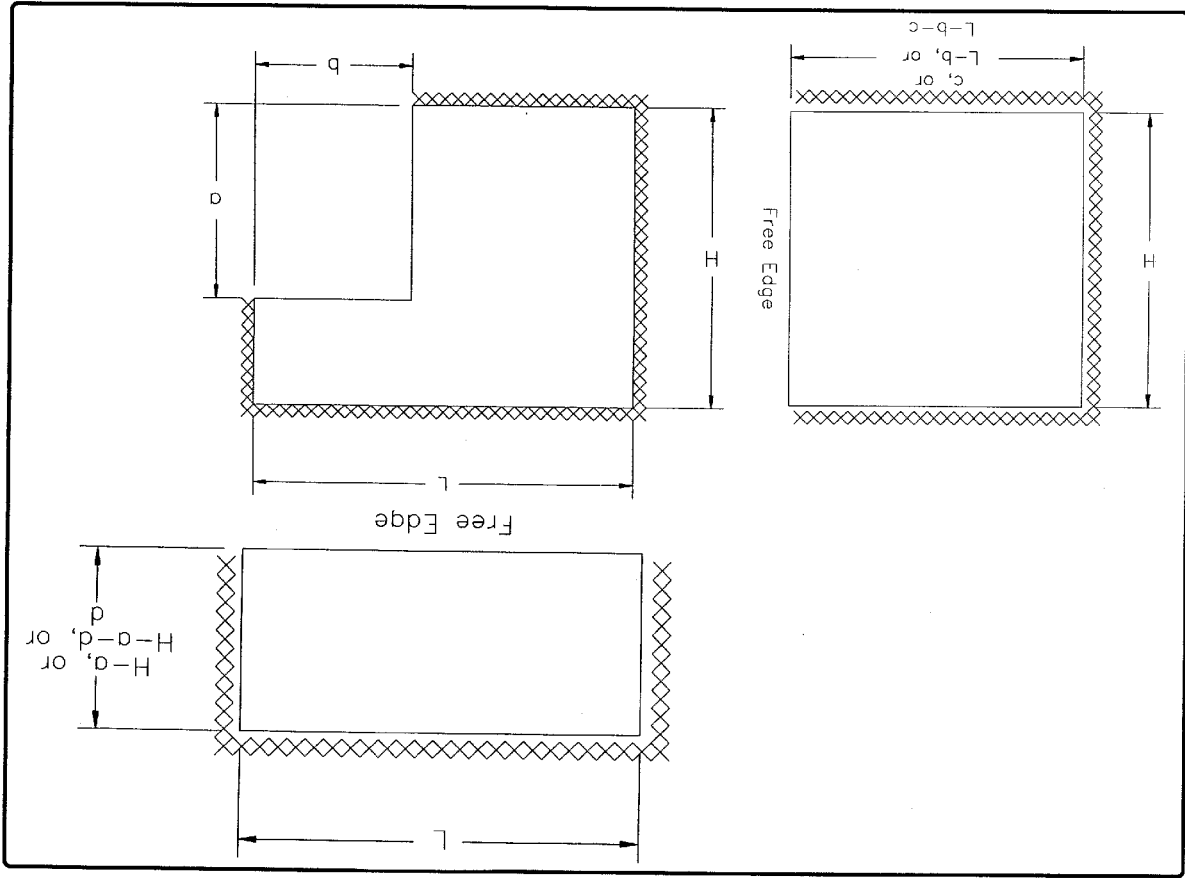
Therefore:

$$H = 15 \text{ ft}$$

$$L - b = 25 \text{ ft} - 7.5 \text{ ft}$$

$$= 17.5 \text{ ft}$$





### **Step 6: Compute Stiffness of Slab With Covered Opening**

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**Difference between  $K_{e1}$  and  $K_{e2}$ :**

$$\begin{aligned} K_{e1} - K_{e2} &= 2053.9 - 1223.5 \\ &= 830.4 \text{ psi-in.} \end{aligned}$$

$$\begin{aligned} K_e &= K_{e1} - (K_{e1} - K_{e2}) (a/H) \\ &= 2053.9 - 830.4 (0.6667) \\ &= 1500.3 \text{ psi-in.} \end{aligned}$$

